LA lecture 3 on Linear equation systems ++

First: what do we have?

- Vectors:
  - Definition(s). Scale, add, dot.
  - Geometric interpretation. (The budget hyperplane!)
- Matrices:
  - Definition(s). Scale, add, multiply, transpose.
  - Vectors as matrices. Matrices as composed by vectors.
- Q: Do we have any geometry that makes intuition simpler ...?
  - Maybe? To follow: slides 2–3 may or may not help you.
    The geometry interpretation is "optional" (the algebra is not!)

To follow today: Linear equation systems.

- (elementary row) operations on matrices

Post-lecture update: "Example 3" was omitted, as indicated – but left in these notes, and the final examples were done in lecture 4, after which a couple of sentences were added at the end of slide 26. Bugs fixed.

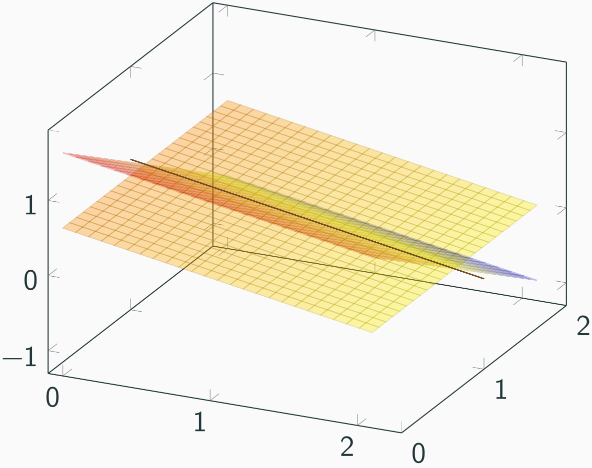
LA lecture 3: Budget constraint as linear equation

- Suppose there are n goods in the economy, and you are about to choose $\mathbf{x} \in \mathbb{R}^n$ to consume[1].
- The budget constraint $\mathbf{p} \cdot \mathbf{x} = \beta$ will remove one degree of freedom from your choice; if[2] $p_1 \neq 0$, then once $x_2, \ldots, x_n$ are chosen, x_1 will be pinned down to $\frac{1}{p_1}\left[\beta - p_2 x_2 - \cdots - p_n x_n\right]$.
- What if someone imposes another linear constraint $\mathbf{r} \cdot \mathbf{x} = \gamma$ on you?
  - Next slide: $n = 3$; think of a budget $(1, 3, 3) \cdot (x, y, z) = 5$ (the plane "with blue edge"!) and throw in another linear equation.

[1] assuming you can actually consume negative amounts

[2] what if $p_1 = 0$? Then choose some other non-free good to solve for. Works unless $\mathbf{p} = \mathbf{0}$ ... in which case, what happens?

Visualization: assume $n = 3$. Two eq's:



Two planes: $(1,\ 0,\ 3) \cdot (x,\ y,\ z) = 2$ and $(1,\ 3,\ 3) \cdot (x,\ y,\ z) = 5$.

The intersection is the line $(x,\ y,\ z) = (2,\ 1,\ 0) + t(-3,\ 0,\ 1)$.

LA lecture 3: Linear equation systems

Requiring (x, y, z) to belong to (both simultaneously!) the two planes $(1,\ 0,\ 3) \cdot (x,\ y,\ z) = 2$ and $(1,\ 3,\ 3) \cdot (x,\ y,\ z) = 5$ and $(1,\ 0,\ 3) \cdot (x,\ y,\ z) = 2$, is the same as imposing the *system of two linear equations:* $\quad x + 3z = 2$ & $x + 3y + 3z = 5$. Or, written on matrix form: $\begin{pmatrix} 1 & 0 & 3 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

- Solution with one *degree of freedom.* (A line.)
- If there were another third equation: *Would typically* eliminate that degree of freedom and pin down one point where that third plane is hit by the line.
    - ... but not necessarily so. E.g., if the third eq. is $y = c$: If $c \neq 1$: impossible! If $c = 1$: still the same line.

Next slide: general theory

A **linear equation system** for an unknown $n \times p$ matrix $\mathbf{X}$ is (or can be written as) $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A}$ is $m \times n$, $\mathbf{B}$ is $m \times p$

- Such an eq. system has either no solution, unique (i.e. precisely one) solution, or infinitely many solutions!
  - Case $\mathbf{B} = \mathbf{0}_{m\times p}$ – a so-called *homogeneous* equation system: Then, there always is at least one solution, $\mathbf{X} = \mathbf{0}_{n\times p}$.
  - (If there are two distinct solutions, $\mathbf{X}$ and $\mathbf{Y}$, then any $\mathbf{Z} = \mathbf{X} + t(\mathbf{Y} - \mathbf{X})$ also solves: $\mathbf{AZ} = \mathbf{AX} + t(\mathbf{AY} - \mathbf{AX}) = \mathbf{B} + t(\mathbf{B} - \mathbf{B})$, *OK*.)
- Exam: You can be asked to "solve". That means: Find *all* solutions, or show that none exists.
- Exam: You can be asked, e.g. *"Does the equation system have zero, one or more than one solution?"* That does not ask you to solve!
  - System might depend on parameter c. Question type: *"For what $c \in \mathbb{R}$ does the system $\mathbf{A}_c\mathbf{x} = \mathbf{b}_c$ have unique solution?"*

(cont'd next slide)

LA lecture 3: Linear eq. systems – theory / degrees of freedom

Last sub-item had minuscle $\mathbf{x}$ and $\mathbf{b}_c$ – i.e. column vectors, $p = 1$:

- Exam/syllabus: if $p > 1$, so $\mathbf{X}$ and $\mathbf{B}$ are *not* (column) vectors, then:
  - You will not be asked to solve for *infinitely* many solutions. You will not be asked for degrees of freedom (see below). The rest of the previous slide you should know, though.

In the following: Unless the capital $\mathbf{B}$ is explicitly used, assume $p = 1$ and consider $\mathbf{Ax} = \mathbf{b}$.

Definition: Solution with d *degrees of freedom* means:
$d = 0$: Unique solution.
$d \in \mathbb{N}$: Infinitely many solutions, such that there is some selection of d variables that can be chosen freely, and then, the remaining $n - d$ variables are determined uniquely by the system.

LA lecture 3: Linear equation systems – vs scalar $\alpha x = \beta$

The case of one single equation in a single unknown? $\alpha x = \beta$.

- Square coefficient matrix :-) (1 × 1 is "square")
  $\leftrightarrow$ as many equations as unknowns, cf. the counting rule – which is *only a "rule of thumb", not logically valid!*
  - If α^{-1} exists (i.e. if $\alpha \neq 0$): Unique solution.
  - If α^{-1} does not exist: $0x = \beta$ either has no solution (if $\beta \neq 0$) or solution with one degree of freedom.

Q: What properties generalize from $\alpha x = \beta$ to $\mathbf{AX} = \mathbf{B}$, and how?

- $\mathbf{A}$ alone determines whether there is unique solution or not.
- If not unique: None or infinitely many; one must consider both $\mathbf{A}$ and $\mathbf{B}$ to determine (i) whether none or infinitely many; and (ii) if infinitely many: how many degrees of freedom.
- If $\mathbf{A}$ is square: unique solution iff $\mathbf{A}$ has an *inverse* $\mathbf{M}$ such that $\mathbf{MA} = \mathbf{I}$: more tomorrow!
- If $\mathbf{A}$ not square: start to solve! Math 2 has no other tools.

LA lecture 3: Linear equation systems – example of $\mathbf{A}\mathbf{x} = \mathbf{b}$

Example: Back to $x + 3z = 2$ & $x + 3y + 3z = 5$. Subtract eq's to get $3y = 3$ and $x + 3z = 2$, one degree of freedom:

- Either choose $z = t$; then x will be given as $y = 2 - 3t$;
- Or, choose $x = s$; then z will be given as $z = (2 - s)/3$.
- *Note:* y cannot be chosen freely. All solutions have $y = 1$.

What did I just do to solve ... ?

$$\begin{array}{ccccccl} x & & & + & 3z & = 2 & \quad -1 \\ x & + & 3y & + & 3z & = 5 & \leftarrow + \end{array} \quad \Leftrightarrow \quad \begin{array}{ccccccl} x & & & + & 3z & = 2 \\ 0 & + & 3y & + & 0 & = 3 \end{array}$$

Matrices: $\begin{pmatrix} 1 & 0 & 3 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Next up: write as $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 1 & 3 & 3 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right)$.

(Scaled the last by $1/3$, then it says "$y = 1$".)

Lots of phrases coming up, some "not exam relevant[3]":

Definition: The *augmented coefficient matrix* of the equation system $\mathbf{AX} = \mathbf{B}$, is the matrix $(\mathbf{A}\big|\mathbf{B})$ composed by stacking up $\mathbf{B}$ to the right of $\mathbf{A}$. (Like on previous slide.)

- The $\Big|$ is not "completely standard" notation, but recommended to keep left-hand side from right-hand side.

More terminology follows:

[3] At the exam, you will not be asked "what are elementary row operations?" nor "what is row-echelon form?" (indeed, you can say *"staircase"* if you like) – but you need to be able to get there using those operations. And teaching needs some language ...
You will not be asked "what is Gaussian elimination?", *but you could be asked, e.g.:* "Solve [...] by Gaussian elimination", and then you must use that method – which means you must know which method it refers to.

(reduced) row-echelon form: A matrix is on row-echelon form if:

every row has a *leading one*
i.e.: first nonzero element $= 1$
(leading ones: green)
&: all zeroes below leading 1s

$$\begin{pmatrix} 1 & ? & ? & ? & ? & \dots \\ 0 & 1 & ? & ? & ? & \dots \\ 0 & 0 & 0 & 1 & ? & \dots \\ \vdots & & \dots & & & \end{pmatrix}$$

Reduced row-echelon: if furthermore all elements *above* leading ones, are zero as well: the blue question marks should be 0.

Good for: An augmented coefficient matrix on row-echelon form is "easy to solve bottom–up". Example: $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right)$

Third row says $x_3 = 7$. Second says $x_2 + 4x_3 = 3$, we solve for $x_2 = 3 - 4 \cdot 7 = -25$. First row says $x_1 + x_2 + 3x_3 = 4$, and so $x_1 = 4 + 25 - 21 = 8$.

Example: $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Even easier! The leading 1 for the third row, belongs to the RHS – and so the third equation says $0 = 1$. No solution!

Reduced row–echelon form "has already been solved bottom–up".

Example: $\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 6 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 7 \end{array}\right)$ (Three eq's, five unknowns)

Leading 1s are in columns, 1, 2, 4. So x_1, x_2, x_4 will be determined once the others (x_3, x_5) are chosen freely.

(Indeed, x_5 does not enter at all.) Choose $x_3 = s$, $x_5 = t$, and write out: $x_4 = 7$, $x_2 = 3 - 6s$, $x_1 = 4 - 2s$.

We want a recipe for converting an eq. system to row-echelon:

Gaussian elimination

- An algorithm to solve linear equation systems, by
  - interchanging equations
  - scaling equations by nonzero numbers
  - adding (a scaling of) an equation to another – or subtracting

  (Subsumes your old "isolate and insert" method!)
- These operations can be performed on the equation system, or on the augmented coefficient matrix[4,5].
- Yields the full solution ("none" if none exists).
- Exam: if asked to "solve by Gaussian elimination", you shall

[next slide]

[4]On the matrix, they are called "elementary row operations". I will use that term, you only need to know the recipe.

[5]For a homogeneous system, you can omit the RHS and do the operations on $\mathbf{A}$. Why? Because the RHS will *remain* zero throughout the algorithm.

Gaussian elimination:

- Exam: if asked to "solve by Gaussian elimination", you shall
  - use the above operations until you can conclude that no solution exists (then stop!) OR until row-echelon form, & then:
  - from row-echelon form on, you can choose whether to solve bottom–up, or to continue Gaussian elimination until reduced row-echelon form ...
  - although, if the unknown is not a column vector ($\mathbf{AX} = \mathbf{B}$), then eliminate until reduced row-echelon form. Should that occur on the exam, you will either arrive at $(\mathbf{I} \mid \mathbf{M})$ so that $\mathbf{X} = \mathbf{M}$ – or at no solution.

  You are not required to apply the following cookbook "in order"; as long as you apply the same operations, you can take shortcuts if you find them.

Gaussian elimination cookbook:

- If at any stage in the below algorithm you get a *"zero equal to nonzero"* equation: declare *"no solution"* and stop. Done!
- Any *zero row* is a "$0 = 0$" equation: *Delete it.*
- Any "variable not appearing" is free if solution exists.

Start at the top–left of $(\mathbf{A} \mid \mathbf{B})$: first equation and first variable.

Step 0: If the first column is the null column vector, then move one column to the right; repeat if necessary.

Step 1: First variable: get a nonzero coefficient in the first eq. by interchanging rows if necessary.

Step 2: Scale the first row by 1/ that coefficient: gets a leading 1.

Step 3: *eliminate* all nonzeroes underneath this 1 by adding a scaling of the first row.

[continued]

LA lecture 3: Linear eq.; Gaussian elimination cookbook.

After Step 3, you have a leading 1 and all zeroes underneath it. Now you can declare that row (and all above) done, keep them as they are – at least until step 5.

Step 4: Start over at step 0 on the block "to the south–east": i.e., that starts with the next row & next column. Illustration:

$$\left(\begin{array}{ccc|c} 1 & ? & ? & ? \\ \mathrm{Z} & & & \\ \mathrm{E} & & & \\ \mathrm{R} & & & \\ \mathrm{O} & & & \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccccc|c} 1 & ? & ? & ? & \dots & ? \\ 0 & 1 & ? & ? & \dots & ? \\ \mathrm{Z} & \mathrm{E} & & & & \\ \mathrm{R} & \mathrm{O} & & & & \\ \mathrm{E} & \mathrm{S} & & & & \end{array}\right)$$

Step 4: IOW, start at step 0 on the green block – with "first row" as "first in the green block". *Repeat until no rows left.*

Step 5: Solve bottom–up. Or, eliminate "upwards", eliminating everything *above* leading ones. Then read off the solution.

Example 1 w/o matrix notation: Recall the example $x + 3z = 2$ & $x + 3y + 3z = 5$; introduce another equation $x + y + z = 0$ for three eq.'s in three unknowns. Write out (aligned vertically):

$$\begin{aligned} x \phantom{{}+3y} + 3z &= 2 && \text{(I)} \\ x + 3y + 3z &= 5 && \text{(II)} \\ x + y + z &= 0 && \text{(III)} \end{aligned}$$

- Steps 0–2: Lucky us, the top–left coefficient is already 1.
- Step 3: Eliminate the *other* x-coefficients by adding a multiple of the first equation. In this case: -1 of (I) to (II) and (III)

$$\begin{aligned} &\text{keep this} && x \phantom{{}+3y} + 3z = 2 && \text{(I)} \\ &\text{subtracting (I), we get} && 3y \phantom{{}-2z} = 5 - 2 && \text{(II')} \\ &\text{subtr. (I) from (III) as well} && y - 2z = 0 - 2 && \text{(III')} \end{aligned}$$

Now we consider the section $\begin{smallmatrix} 3y & = & 3 \\ y - 2z & = & -2 \end{smallmatrix}$, leaving (I) as-is.

Equation 2 says $3y = 3$, so steps 0–1 done. Step 2: Scale by $1/3$ to get a leading 1, and the eq. system.s

done 'til step 5:	$x \quad + 3z = 2$	(I)
scaled by 1/3	$y \quad = 1$	(II")
(nothing done here yet)	$y - 2z = -2$	(III')

Step 3: eliminate the y-coefficient from (III') by adding (-1) times (II"). Third eq. then becomes $-2z = -3$. We have:

$$\begin{aligned} x \quad + 3z &= 2 && \text{(I)} \\ y \quad &= 1 && \text{(II")} \\ -2z &= -3 && \text{(III")} \end{aligned}$$

Last equation left: step 2, scale by $(-1/2)$ to get $z = 3/2$ (III'''). Now, each eq. has a leading one. We can either solve bottom–up; $z = 3/2$, $y = 1$ and $x = 2 - 3z = -5/2$. Or, eliminate upwards: add (-3) of (III''') to (I) to eliminate the "$3z$".

On to matrix notation[6] – and a system *depending on a constant:*

Example 2: $\left(\begin{array}{cccc|c} t & 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & t \\ 2 & 3 & 0 & 4 & 5 \end{array}\right).$

- Here, t is *not* an unknown. This is one equation system for each value of t.
- Scaling by $1/t$? Then you have to split between cases $t = 0$ and $t \neq 0$. Lots of work ... that we can avoid for a while.
- Better: Move any division by t "so far into the future as we can". Reordering the one on the RHS is not so bad, we shall not divide by it. So get the first row all the way down!
- Suggestion: get the second row first (no scaling, no fractions) – but if you prefer, you can just interchange rows 1 and 3.

[6] Exercises: write (I)–(III) of Example 1 on matrix form, do the same operations, and compare; then do Example 2 *without* matrices and compare.

LA lecture 3: Linear eq. systems; Gaussian elimination ex. 2

Notation: $\sim$ for “represents equivalent equation system”.

Reordering: $\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & t \\ 2 & 3 & 0 & 4 & 5 \\ t & 1 & 0 & 2 & 3 \end{array}\right) \begin{array}{l} -2 \quad -t \\ + \\ + \end{array}$

Now eliminate 2 and t (“step 3”), using the operations indicated:

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & t \\ 0 & -1 & 0 & -2 & 5-2t \\ 0 & 1-2t & 0 & 2-3t & 3-t^2 \end{array}\right) \begin{array}{l} \\ \mid \cdot (-1) \\ \\ \end{array}$$

First row and column done, go on with the $\begin{array}{ccc|c} -1 & 0 & -2 & 5-2t \\ 1-2t & 0 & 2-3t & 3-t^2 \end{array}$ block; the “$\cdot(-1)$” is step 1 and gets a leading 1 in row 2:

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & t \\ 0 & 1 & 0 & 2 & 2t-5 \\ 0 & 1-2t & 0 & 2-3t & 3-t^2 \end{array}\right) \begin{array}{l} \\ -(1-2t) \\ + \end{array}$$

(If we want to end up with reduced row-echelon form, we could simultaneously subtract 2 of row 2 from row 1!)

LA lecture 3: Linear eq. systems; Gaussian elimination ex. 2

Last row becomes $(0,\ 0,\ 0,\ 2{-}3t{-}2{+}4t \mid 3{-}t^2{-}2t{+}5{+}4t^2{-}10t)$:

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & t \\ 0 & 1 & 0 & 2 & 2t-5 \\ 0 & 0 & 0 & t & 3t^2-10t+8 \end{array}\right)$$

Now on to the last row, and here we have a "step 0": from the "block" $(0,\ t,\ 3t^2-10t+8)$, we move one step to the right. Then the t forces us to split into cases $t=0$ vs. $t \neq 0$. But that is much easier now than had we done so at the very beginning:

- Case $t=0$: Last eq. says $0=8$. *No solution!*
- Case $t \neq 0$: now we can divide by t, and the last row becomes $(0,\ 0,\ 0,\ 1,\ 3t-10+8/t)$ – row-echelon! Solve bottom–up with $x_3 = 3t-10+8/t$ (or eliminate upwards, if you prefer).
    - In the end, make sure you *do not* use the letter t for degree of freedom – in this problem, that is already used.
    - In fact, x_3 does not enter the system! You could already at the very beginning conclude "x_3 free if a solution exists at all".

LA lecture 3: Linear eq. systems; Gaussian elimination ex. 3

Example 3 (bigger) – this was skipped in the interest of time. Review it if you like – I think it is explained thoroughly

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 4 & 2 & 4 & 3 & 4 \\ 2 & 2 & 2 & 2 & 4 \\ 3 & 4 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

No "step 0", as there is some nonzero in the first column. Step 1: to get a nonzero in element (1,1), interchange row 1 with e.g. 3. In step 2, scale by $1/2$, and then step 3 is indicated:

$$\sim \left(\begin{array}{ccccc|c} 2 & 2 & 2 & 2 & 4 & 2 \\ 4 & 2 & 4 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 3 & 4 & 3 & 5 & 6 & 1 \end{array}\right) \begin{array}{l} \mid \cdot 1/2 \\ \\ \\ \\ \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 4 & 2 & 4 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 3 & 4 & 3 & 5 & 6 & 1 \end{array}\right) \begin{array}{ll} -4 & -3 \\ + & \\ & \\ & + \end{array}$$

LA lecture 3: Linear eq. systems; Gaussian elimination ex. 3

Step 3, the elimination, slowly: The "−4" is what it takes to eliminate element # (2,2). The "−3" eliminates element # (4,2). Element # (3,2) is already zero. The first row is kept!

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 4 & 2 & 4 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 3 & 4 & 3 & 5 & 6 & 1 \end{array}\right) \begin{array}{l} \urcorner^{-4}\ \urcorner^{-3} \\ \leftarrow^{+} \\ \\ \leftarrow^{+} \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & -2 & 0 & -3 & -4 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & -2 \end{array}\right)$$

Now we are done with the first two columns, and the first row. Keep these, and return to step 0 on the block $\begin{array}{ccc|c} -2 & -3 & -4 & -1 \\ 0 & 1 & 2 & 4 \\ 1 & 2 & 0 & -2 \end{array}$.

Nothing to do in steps 0, 1; for step 2, scale by $-1/2$, and then:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 0 & -2 \end{array}\right) \begin{array}{l} \\ \urcorner^{-1} \\ \\ \leftarrow^{+} \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1/2 & -2 & -5/2 \end{array}\right)$$

Return to step 0 on the block $\begin{smallmatrix} 0 & 1 & 2 \\ 0 & 1/2 & -2 \end{smallmatrix} \left| \begin{smallmatrix} 4 \\ -5/2 \end{smallmatrix} \right.$. Here is where step 0 is used: the first column of that block is all zeroes; move one step to the right and consider $\begin{smallmatrix} 1 & 2 \\ 1/2 & -2 \end{smallmatrix} \left| \begin{smallmatrix} 4 \\ -5/2 \end{smallmatrix} \right.$. No steps 0/1/2; step 3: subtract half of the third (= first of these two) from the last:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1/2 & -2 & -5/2 \end{array}\right) \begin{array}{l} \\ \\ \rceil^{-1/2} \\ \leftarrow\!\rfloor_{+} \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3/2 & 2 & 1/2 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & -3 & -9/2 \end{array}\right)$$

Finally, consider the last row: "Step 1", scale by $-1/3$ to get the row $(0\ 0\ 0\ 0\ 1 \mid 3/2)$, obtaining the staircase ("row-echelon form")

$$\left(\begin{array}{ccccc|c} \color{green}{1} & 1 & 1 & 1 & 2 & 1 \\ 0 & \color{green}{1} & 0 & 3/2 & 2 & 1/2 \\ 0 & 0 & 0 & \color{green}{1} & 2 & 4 \\ 0 & 0 & 0 & 0 & \color{green}{1} & 3/2 \end{array}\right)$$

. (Note: Ever row has a "leading 1".)

x_3 does *not* correspond to a leading 1, and will be free.

Step 5! The "solve bottom–up" alternative is straightforward, once we have put $x_3 = t$ (free)? Last row says $x_5 = \frac{3}{2}$. Row 3 says $x_4 + 2x_5 = 4$, so $x_4 = 4 - 3 = 1$. Row 2: $x_2 = \frac{1}{2} - \frac{3}{2}x_4 - 2x_5$ $= -4$. And finally the first row: $x_1 = 1 - x_2 - x_3 - x_4 - 2x_5$; here x_3 enters! Inserting, we get $x_1 = 1 + 4 - t - 1 - 3 = 1 - t$. Solution: $\mathbf{x} = (1 - t,\ -4,\ t,\ 1,\ \frac{3}{2})'$.

Step 5, the "eliminate upwards" alternative: exercise!

LA lecture 3: Ex. 4: $\mathbf{AX} = \mathbf{B}$, eliminated to reduced row-echelon

Example 4: $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 3 & 3 & 5 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|cc} 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 4 & 3 & 2 \\ 3 & 3 & 5 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & -1 & -2 & -6 \end{array}\right)$$

"Cookbook" says change sign and subtract; here, I add first.

Afterwards also changing sign on row 3, we will get:

$$\sim \left(\begin{array}{ccc|cc} 1 & 1 & 0 & -3 & -10 \\ 0 & 1 & 0 & -2 & -12 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right) \sim \left(\begin{array}{c|cc} & -1 & 2 \\ \mathbf{I}_3 & -2 & -12 \\ & 2 & 6 \end{array}\right)$$

The latter says: $\mathbf{I}_3\mathbf{X} = \begin{pmatrix} -1 & 2 \\ -2 & -12 \\ 2 & 6 \end{pmatrix}$.

LA lecture 3: Ex. 5 & 6: $\mathbf{AX} = \mathbf{B}$, eliminated to ... ?

Example: What about $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 3 & 3 & 5 \end{pmatrix} \mathbf{X} = \mathbf{I}_3$? (Left as exercise!)

Example: $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \end{pmatrix}$. (Changed: element a_{31}.)

$$\left(\begin{array}{ccc|cc} 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 4 & 3 & 2 \\ 1 & 3 & 5 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 4 & 0 & -2 \end{array}\right)$$

Last row becomes $(0\ 0\ 0 \mid -2\ -2)$. No solution!

(If you want it related to the "cookbook": scale the latter to $(0\ 0\ 0 \mid 1\ 1)$ and see that the leading "1" belongs to the right-hand side!)

Again: in this course, when $\mathbf{B}$ is not a vector, Gaussian elimination all the way would either lead to $(\mathbf{I} \mid \mathbf{M})$ (so that $\mathbf{X} = \mathbf{M}$) or to some contradiction. Other cases: only if $\mathbf{B} = \mathbf{b}$, a vector.

LA lecture 3: Gaussian elimination – "finding shortcuts"

Cookbook "works", while an arbitrary selection of operations "might run in circles". But, sometimes, we can speed up.

Example: $$\left(\begin{array}{cccc|c} \multicolumn{5}{c}{\text{[some row(s)...]}} \\ 4 & 8 & 12 & 16 & c \\ 2 & 4 & 6 & 8 & 1 \\ \multicolumn{5}{c}{\text{[... more rows]}} \end{array}\right) \begin{array}{l} \leftarrow + \\ -2 \end{array}$$

Two rows with proportional left-hand sides; consider handling those first. No need to downscale to leading 1.

- Either: no solution (if $c \neq 2$) –
- or if $c = 2$: a redundant row that *can be deleted.*

Here, splitting in cases early is A Good Thing, because you can conclude in one case – and in the other, you can insert $c = 2$ everywhere else in the system.

(You prefer numbers over constants that may or may not be zero, don't you? Remember: *never divide by zero!*)

LA lecture 3: Gaussian elimination – “finding shortcuts”

Example: (exercise: try instead to subtract 11 of the first from #2 and #4 ...)

$$
\begin{array}{l}
\left(\begin{array}{cccc|c}
0 & 1 & 2 & 3 & 4 \\
111 & 122 & 133 & 144 & 155 \\
222 & 333 & 444 & 555 & 666 \\
222 & 233 & 244 & 255 & 266
\end{array}\right)
\end{array}
\begin{array}{l}
\\
\mid \cdot 2 \leftarrow + \\
\leftarrow + \\
-1 \quad -1 \text{ \textit{after} having scaled \#2}
\end{array}
$$

Avoiding fractions: scale by 2 rather than by $1/111$ (resp. $1/222$).

$$
\sim \left(\begin{array}{cccc|c}
0 & 1 & 2 & 3 & 4 \\
0 & 244-233 & 266-244 & 288-255 & 310-266 \\
0 & 100 & 200 & 200 & 400 \\
222 & 233 & 244 & 255 & 266
\end{array}\right)
$$

3rd row – and second $(0,\ 11,\ 22,\ 33,\ \mid 44)$ – proportional to first row. Delete! Subtract 233 of first from last, then it becomes

$$
\begin{pmatrix} 222 & 0 & -222 & -444 & \mid -666 \end{pmatrix} = 222 \begin{pmatrix} 1 & 0 & -1 & -2 & \mid -3 \end{pmatrix}
$$

Solution: Choose x_3 and x_4 free, and then $x_2 = 4 - 2x_3 - 3x_4$ while $x_1 = x_3 + 2x_4 - 3$.