LA lecture 4: linear eq. systems, (inverses,) determinants

Yesterday:

- Linear equation systems
  - Theory
  - Gaussian elimination

To follow today:

- Gaussian elimination – leftovers
- A bit about the inverse:
  - Definition
  - If $\mathbf{A}$ has an inverse, call it $\mathbf{A}^{-1}$, then the equation system $\mathbf{AX} = \mathbf{B}$ has precisely one solution, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
  - We can *calculate* $\mathbf{A}^{-1}$ by eliminating $(\mathbf{A}|\mathbf{I})$ down to $(\mathbf{I}|\mathbf{M})$
  - ... or by a formula involving *determinants*.
- Determinants, then.
  - "Definition" [loosely – what you need to know]
  - Properties
  - Two ways to calculate.

Post-lecture update: Bugfix extravaganza.

LA lecture 4: Gaussian elimination leftovers

[End of previous slideset, see that one.]

Back to the scalar equation $\alpha x = \beta$:

- If α^{-1} exists, then the only solution is $\alpha^{-1}\beta$.
- That "α^{-1}" we often write $1/\alpha$: we can divide by numbers.
- Matrices: cannot divide, but sometimes we have an "$\mathbf{A}^{-1}$" and can multiply.

Definition: Given $\mathbf{A}$. If there exists $\mathbf{M}$ such that $\mathbf{MA} = \mathbf{AM} = \mathbf{I}$, then we call $\mathbf{M}$ *the inverse of* $\mathbf{A}$ and write $\mathbf{A}^{-1} = \mathbf{M}$.

- $\mathbf{A}$ must necessarily be square. $\mathbf{M}$ must be of same order.
- **Fact:** $\mathbf{A}$ cannot have more than one inverse.
- **Fact:** If $\mathbf{A}$ is square, then you need only check one of $\mathbf{AM} = \mathbf{I}$ or $\mathbf{MA} = \mathbf{I}$, because then the other holds automatically:

  For $n \times n$ matrices, $\quad \mathbf{AM} = \mathbf{I} \quad$ holds if and only if $\quad \mathbf{MA} = \mathbf{I}$

(Not at all obvious! True because $n < \infty$ ...)

Once we point out that $\mathbf{A}$ is square, we need only calculate one of the products. Examples:

Example: Show that $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}^{-1} = t \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ for some $t \in \mathbb{R}$.

Solution: Multiply: $t \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = t \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = -2t\mathbf{I}$.
True when $t = -1/2$.
Or, just as good: $-\frac{1}{2} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \mathbf{I}$.

Example: If $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(Fact: If $ad - bc = 0$, then no inverse exists.
For later: $ad - bc$ is the *determinant* of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.)

Once we point out that $\mathbf{A}$ is square ... cont'd:

Fact: If $\mathbf{A}$ is square, then solving $\mathbf{AX} = \mathbf{I}$ for $\mathbf{X}$, yields $\mathbf{X} = \mathbf{A}^{-1}$ if it exists (and no solution if it doesn't.))

Why? Solving, we do get $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I}$ – iff that exists.
Then we do not need to verify that $\mathbf{XA} = \mathbf{I}$. (Since $\mathbf{A}$ is square.)

Furthermore:
Fact: A matrix $\mathbf{A}$ has an inverse *iff*

$\mathbf{A}$ is square AND has nonzero *determinant*

What? A special function that takes a *square* ($n \times n$) matrix as input, and returns a number.

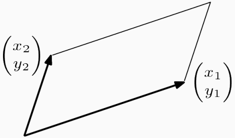
History: used for linear eq. systems before matrices! Nine Chapters of Mathematical Art, China, $\approx$ 200 BCE ...

Does that number mean anything? It has an area/volume/ hypervolume interpretation you do not need to worry about.
For the curious: diagram next page.

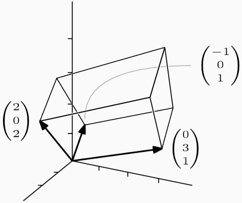
More important in this course: if $\mathbf{A}$ has nonzero determinant, then $\mathbf{A}^{-1}$ exists and the system $\mathbf{Ax} = \mathbf{b}$ (which is n equations in n unknowns) *obeys the counting rule.*

LA lecture 4: The determinant: geometry (OPTIONAL!)

Geometry (OPTIONAL). If it helps your intuition – or curiosity: The determinant of $\mathbf{A}$ equals $\pm$ *the n-dimensional hypervolume of the n-dim. parallelogram spanned by the columns of $\mathbf{A}$.* Or by the rows of $\mathbf{A}$. Illustration, $n = 2$ and $n = 3$:



Determinant $= \pm$ the area, resp. the volume



Source: Wikibooks "Linear Algebra/Determinants as Size Functions", diagrams derived by Nicholas Longo from Jim Hefferon's open-source linear algebra book, license: CC-BY-SA-2.5.

A square matrix $\mathbf{A}$ has an inverse as long as that "hypervolume" does not collapse to zero:

$n = 2$ as long as the columns are not on the same line. $n = 3$: as long as the columns are not in the same plane. Etc.

LA lecture 4: The determinant

Notation: $\texttt{det}\,\mathbf{A}$ or $|\mathbf{A}|$ or "bar delimiters": $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for $\texttt{det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Beware, it is not an absolute value – and could be negative.

How to define? There are $n! = n(n-1)\cdots 2\cdot 1$ ways to select n elements with precisely one from each row and column. For each selection, multiply the elements. Switch sign on half the products, and add up. The "sign switching" follows this rule:

- The selection of all the main diagonal elements: do not switch.
- Every time two rows are interchanged: switch sign.

Example: 2×2 and 3×3, the "NE/SW" diagonal. Resp.:
$\begin{pmatrix} \cdot & \star \\ \star & \cdot \end{pmatrix}$ and $\begin{pmatrix} \cdot & \cdot & \star \\ \cdot & \star & \cdot \\ \star & \cdot & \cdot \end{pmatrix}$. Would be the main diagonal if the 1st and last row were interchanged. Precisely one interchange $\leftrightarrow$ precisely one sign switch. Put a negative in front.

You will likely not use the definition except a very few cases:

- 1×1: From the definition, the determinant is the element. (Again: Not any absolute value! Not even writing the bars here ...)
- 2×2: From the definition, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Maybe also for the following, although they will also follow from the rules to follow next slides:

- 3×3: Here you can also read it from the book. Or: "Sarrus rule" (only valid for 3×3, not to be covered here).
- *Triangular* matrices: upper (resp. lower) triangular $\leftrightarrow$ all zeroes below (resp. above) the main diagonal: *Determinant = the product of the main diagonal elements.* (Why? Every other selection of precisely one element from each row/col., contains a zero.)

The *cofactor* C_{ij} of element (i, j) of $\mathbf{A}$, is formed by

- Deleting row i and column j from $\mathbf{A}$ (that is, the row and column of said element)
- Calculating the determinant of the rest $((n-1) \times (n-1))$
- Multiplying by $(-1)^{i+j}$.

Example: The cofactor C_{23} of element $(2, 3)$ of $\begin{pmatrix} 3 & 9 & -2 \\ -1 & 5 & \boxed{\mathbf{7}} \\ -11 & 4 & 7 \end{pmatrix}$ is (note how nothing magenta appears):

$(-1)^5 \begin{vmatrix} 3 & 9 \\ -11 & 4 \end{vmatrix} = -(12 - (-99)) = -111.$

The "$(-1)^{i+j}$": chessboard $\begin{pmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ \vdots & & & & \ddots & \end{pmatrix}$

LA lecture 4: The determinant. Cofactor expansion:

Fact: For any $i = 1, \ldots n$, we have $|\mathbf{A}| = a_{i1}C_{i1} + \ldots + a_{in}C_{in}$.
IOW, we can calculate the determinant as follows:

- Pick one row number i (and stick to it!)
- For each element in the row, multiply it by its cofactor
- Add up.

This is called *cofactor expansion along the* i*th row.*

Example: $\begin{vmatrix} 3 & 9 & -2 \\ 2 & 5 & -1 \\ -11 & 4 & 7 \end{vmatrix}$ along the 2nd row: $2C_{21} + 5C_{22} + (-1)C_{23}$

Need $C_{21} = (-1)^{2+1} \begin{vmatrix} 9 & -2 \\ 4 & 7 \end{vmatrix} = -(63 - (-8)) = -71$

and $C_{22} =$ [you go ahead!]: $(-1)^{2+2} \begin{vmatrix} 3 & -2 \\ -11 & 7 \end{vmatrix} = -1$

and $C_{23} =$ [from previous slide] $= -111$

Answer: $2 \cdot (-71) + 5 \cdot (-1) - 1 \cdot (-111) = -36$.

LA lecture 4: The determinant. Cofactor expansion:

Fact: For any $j = 1, \dots n$, we have $|\mathbf{A}| = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}$.
That is *cofactor expansion along the* j*th column.*

Example: $\begin{vmatrix} 3 & 9 & -2 & 1 \\ 2 & 5 & -1 & 0 \\ -11 & 4 & 7 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix}$ along the 4th column.

$$1 \cdot (-1)^{1+4} \underbrace{\begin{vmatrix} 2 & 5 & -1 \\ -11 & 4 & 7 \\ 0 & 0 & -1 \end{vmatrix}}_{\text{expand along 3rd row}} + 0 + 0 + 1 \cdot (-1)^{4+4} \underbrace{\begin{vmatrix} 3 & 9 & -2 \\ 2 & 5 & -1 \\ -11 & 4 & 7 \end{vmatrix}}_{=-36,\text{ prev. slide}}$$

$$= \begin{vmatrix} 2 & 5 \\ -11 & 4 \end{vmatrix} - 36 = 8 + 55 - 36 = 27$$

You can choose which row or which column. If there is one with a lot of zeroes, if often pays off to choose that one. (Although, ...)

For $n \times n$ matrices, the following rules apply:

(i) Cofactor expansion applies along any row/column.

(ii) If one row/column is zero, the determinant is zero.

(iii) $|\mathbf{A}'| = |\mathbf{A}|$.

(iv) If you scale one row by α, then you scale the determinant by α. Same with "column".

(v) Interchange two rows of $\mathbf{A}$, and you switch sign on the determinant (but keep the absolute value!). Same with "two columns". But do not try to interchange a row with a column!

(vi) The determinant does not change if a scaling α of a row is added to a *different row.* Same with "column".
   - So if two rows are proportional, then $|\mathbf{A}| = 0$. Or if "two columns are proportional". But not "a row and a column".

(vii) $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$. Beware that both must be square, not just the product.

Note: There is no simple rule for $|\mathbf{A}+\mathbf{B}|$.

Exercises: What is $|t\mathbf{I}_n|$? What is $|t\mathbf{A}|$?

Beware the common error! Answers: t^n resp. $t^n|\mathbf{A}|$.
Why? Scaling *just one* row by t will scale the determinant by t;
and, scaling the entire matrix $\leftrightarrow$ scaling every one of them.
For $n \times n$ matrices, there are n rows.

LA lecture 4: The determinant. Elementary row operations

Recall that in Gaussian elimination, we had three operations on rows. On determinants, they do the following:

- Interchange two rows: switches sign.
- Scale a row: scales the determinant.
- Add a scaling of a row to another: no change.

Example: $\begin{vmatrix} 222 & 333 & 444 \\ 555 & 666 & 777 \\ 1111 & 2222 & 3333 \end{vmatrix} = 111 \cdot 111 \cdot 1111 \cdot \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix}$

and $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 0$

Exam: You can be asked, e.g. "Calculate this *without using cofactor expansion*."

LA lecture 4: The determinant. Elementary column operations

For determinants we can also do the same operations on columns.
But do not apply column operations to equation systems!

Example: $$\begin{array}{ccc} \cdot\,{}^{1}\!/_{11} & \cdot\,{}^{1}\!/_{11} & \\ \multicolumn{3}{c}{\begin{vmatrix} 22 & 33 & 14 \\ 55 & 66 & 13 \\ 11 & 22 & 1 \end{vmatrix}} \end{array} = 11^2 \begin{array}{ccc} -1 & & + \\ -2 & + & \\ \multicolumn{3}{c}{\begin{vmatrix} 2 & 3 & 14 \\ 5 & 6 & 13 \\ 1 & 2 & 1 \end{vmatrix}} \end{array} = 121 \begin{vmatrix} 2 & -1 & 12 \\ 5 & -4 & 8 \\ 1 & 0 & 0 \end{vmatrix}$$

which by cofactor expansion (along what, you think?) equals

$= 121 \begin{vmatrix} -1 & 12 \\ -4 & 8 \end{vmatrix} = -121 \cdot 4 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$ (WHY?). Answer: 484.

(Worried that you might be asked to calculate this w/o cofactor expansion? Not this one, for then you would in the end need to apply the definition on a 3×3, and ... but previous slide: an “easy” zero. Or we could ask that for something that ends up in $\beta \cdot |\alpha\mathbf{I}|$.)