LA lectures 5&6: more eq. systems / determinants / inverses

Have:

- Vectors, matrices, ...
- Linear equation systems: Theory, Gaussian elimination
- More matrices: Determinants and the inverse.

To follow today: More of everything.

- More determinants (put at work!)
- More linear equation systems. Old & new stuff, in particular:
  - Determinants in a problem-type that has been seen frequently on exams
  - Cramér's rule
- More matrices: The inverse
  - A new determinants-based formula, with an example
  - (More rules)

Question for you: anything you need reviewed on Wednesday?

(You requested: another detailed cofactor expansion walkthrough (see slides 27ff). And Sarrus (slides 25–26).)

Example: The ECON2310 seminar 1 macro model could be written

$$\begin{pmatrix} 1 & -1 & -1 & 0 & 1 \\ -c_1 & 1 & 0 & c_1 & 0 \\ -b_1 & 0 & 1 & 0 & 0 \\ -t & 0 & 0 & 1 & 0 \\ -a & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ I \\ T \\ Q \end{pmatrix} = \mathbf{h}, \qquad \mathbf{h} = \hat{\mathbf{h}} + \begin{pmatrix} G \\ -c_2\hat{\imath} \\ -b_2\hat{\imath} \\ 0 \\ 0 \end{pmatrix}$$

(the latter RHS if you like to indicate possible policy instruments – for an example next week).

Unique solution iff coeff. matrix has nonzero determinant.

Cofactor expansion (tip: look for many 0's – say, 2nd column and then indicated in cyan) $-(-1)\begin{vmatrix} -c_1 & 0 & c_1 & 0 \\ -b_1 & 1 & 0 & 0 \\ -t & 0 & 1 & 0 \\ -a & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & 0 & 1 \\ -b_1 & 1 & 0 & 0 \\ -t & 0 & 1 & 0 \\ -a & 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -1 & 1 & 0 \\ -t & 1 & 0 \\ -a & 0 & 1 \end{vmatrix} c_1 + \begin{vmatrix} 1 & -1 & 1 \\ -b_1 & 1 & 0 \\ -a & 0 & 1 \end{vmatrix} = -c_1(1-t) + a + 1 - b_1$$

Observation *(not a typical exam question, but maybe good for "debugging your determinants"):* Wrt. each of the constants a, b and c_1, that determinant on slide 2 is an affine function (i.e. first-order polynomial; note, "Wrt. each" meaning: "does not rule out cross terms"!). That is not a coincidence:

- Generally, the determinant function is affine in each element: Cofactor expansion: $a_{ij}C_{ij}+$ [no a_{ij} elsewhere].
  So for a, b_1 and t: each of these enters (and linearly!) in precisely one element. As a function of t: $\gamma t + \delta$, etc.
- The determinant is linear in each row and in each column.
  So c_1 enters (as a first-order term) *twice* but both are in the same row. Do cofactor expansion along that row, and you see that the determinant must be of the form $\eta c_1 + \epsilon$.

But a parameter that enters in several rows/columns, may have higher order. (Ex.: $|\lambda \mathbf{I}_n - \mathbf{A}|$ is an nth order polynomial in λ.)

Example: (that used to be) frequent exam-type problem:

Consider for each real constant t the matrix and the vector

$$\mathbf{A}_t = \begin{pmatrix} 4 & 3 & 2 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{pmatrix}, \qquad \mathbf{b}_t = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ Questions:

(a) Calculate the determinant of $\mathbf{A}_t$, each $t \in \mathbb{R}$.

(b) For what $t \in \mathbb{R}$ will the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}_t$ have (i) no solution, (ii) precisely one solution, resp. (iii) several solutions?

Note: does not ask to "solve". (But the wording does not *forbid* solving either, so if that is all you know ...)

(a): ("No tempting zeroes" ... but if you want to collect the "t" coefficient easily, how? Or just go ahead calculate.)

$$|\mathbf{A}_t| = -1\begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} + t\begin{vmatrix} 4 & 2 \\ -2 & -4 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix}$$

$$= -(-6) + t \cdot (-16 + 4) + (-6) = \underline{\underline{-12t}}.$$

(Cofactor expansion details: on slide 27, with the "review".)

(b): Unique solution iff $t \neq 0$. For $t = 0$: several solutions (since $\mathbf{b}_0 = \mathbf{0}$, there is at least one, and we know it is not unique.)

Example variant:

What if the first row of $\mathbf{A}_t$ were replaced by $(4444,\ 4443,\ 4442)$? Recall that we can use elementary row operations for determinants, and note that both row 1 and row 3 "decrease by one as we move to the right"; the difference is $4446(1,\ 1,\ 1)$. Indeed, compute

$$\begin{vmatrix} R-2 & R-3 & R-4 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix} \begin{matrix} \leftarrow\!\urcorner^{+} \\ \Big| \\ \lrcorner_{-1} \end{matrix} = \begin{vmatrix} R & R & R \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix} = R\cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix}$$

For the $|\mathbf{A}_t|$ from the previous slide, we had $R = 6$ and got $-12t$. On this, we have $R = 4446$ and get a determinant equal to $\frac{4446}{6} \cdot (-12t) = -4446 \cdot 2t$ (or, just compute it! Exercise: use row operations as long as you can).

But this determinant too, is zero only iff $t = 0$ – and so part (b) would have the same answer for the variation on this page.

Example variant: with $(k\mathbf{A}_t)^k$ in place of $\mathbf{A}_t$. Here, $k \in \mathbb{N}$ a constant.

(a): $|(k\mathbf{A}_t)^k| = |k\mathbf{A}_t|^k = (k^3|\mathbf{A}_t|)^k = \underline{\underline{(-12tk^3)^k}}$. Did you remember the "3"?

(b): Same argument and same answer as the two previous slides!

Example variant: Replace $\mathbf{b}_t$ by $(1, 2, c)'$. $(k = 1)$. Q: How does that change anything, for each $c \in \mathbb{R}$?

A: No change in (a). In (b), eq. system is no longer homogeneous; for $t = 0$, we have no other tools than starting to solve:

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ -2 & -3 & -4 & c \end{array}\right) \begin{array}{l} \leftarrow^{+} \\ \;\; -4 \;\; 2 \\ \leftarrow^{+} \end{array} \sim \left(\begin{array}{ccc|c} 0 & 3 & 7 & -7 \\ 1 & 0 & -1 & 2 \\ 0 & -3 & -7 & c+4 \end{array}\right) \begin{array}{l} \leftarrow^{+} \\ \\ \end{array}$$

First row then says $0 = c - 3$. Infinitely many solutions iff $c = 3$, otherwise no solution. (Is it perfectly clear why?)

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Example variant (mentioned in class for the "slide 4" ex., further tweaked): What if you were also asked for *the degrees of freedom*?

Unique solution, hence no degrees of freedom, if $t \neq 0$ – this goes no matter what the RHS is! Therefore, for case $t = 0$, I change the example to general RHS $= (\alpha,\ \beta,\ \gamma)'$. The previous slide is a special case too. Doing the same operations:

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & \alpha \\ 1 & 0 & -1 & \beta \\ -2 & -3 & -4 & \gamma \end{array}\right) \begin{array}{l} \leftarrow + \\ \cdot(-4),\ \cdot 2 \\ \leftarrow + \end{array} \sim \left(\begin{array}{ccc|c} 0 & 3 & 7 & \alpha - 4\beta \\ 1 & 0 & -1 & \beta \\ 0 & -3 & -7 & \gamma + 2\beta \end{array}\right) \begin{array}{l} \leftarrow + \\ \\ \end{array}$$

First row now says $0 = \alpha - 2\beta + \gamma$. Choose the third variable freely and get one degree of freedom if $0 = \alpha - 2\beta + \gamma$, no solution otherwise. (This still assumes $t = 0$.)

Note: only case with the *maximum* n degrees of freedom, is $\mathbf{0x} = \mathbf{0}$. Here $n = 3$, so for $t = 0$: "One or two, at a glance." (Even: "one at a glance", because $n - 1$ degrees would imply all rows proportional ...)

Example variant: Let (This $\mathbf{M}$ is $\mathbf{A}_0$ from slide 4.)

$$\mathbf{M} = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 0 & -1 \\ -2 & -3 & -4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Calculate $\mathbf{Mv}$.

(b) Why does it follow from part (a) that $|\mathbf{M}| = 0$? You are *not* allowed to calculate the determinant by other means.

(a) [The "easy" part, cheap points. Matrix multiplication yields $\mathbf{0}$.]

(b) $\mathbf{Mv} = \mathbf{0}$ for some *nonzero* vector $\mathbf{v}$; hence the eq. system $\mathbf{Mx} = \mathbf{0}$ has more than one solution, and hence $|\mathbf{M}| = 0$.

Note: If $\mathbf{w}$ solves the *homogeneous* system $\mathbf{Mx} = \mathbf{0}$ then $c\mathbf{w}$ also does, any $c \in \mathbb{R}$ – *but if* $\mathbf{w} = \mathbf{0}$*, this is no more than one vector!* For our nonzero $\mathbf{v}$, all the $c\mathbf{v}$ make infinitely many solutions.

Example (too tricky for exam?) yeah ... consider this optional.

Can determinants say anything about whether there is a solution to the eq. system (4 eq's, 3 unknowns, RHS=column) with

augmented coefficient matrix $\left(\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ -2 & -3 & -4 & 3 \\ 1 & 1 & 1 & c \end{array}\right)$?

Yes. Fact: if the *augmented* coefficient matrix has an inverse, then no solution exists. Note: does not say "iff"!

Why? An exercise in matrix manipulation:

Generally, $\mathbf{Ax} - \mathbf{b}$ can be written as $\mathbf{Mz}$, where $\mathbf{M} =$ the augmented coefficient matrix $(\mathbf{A}|\mathbf{b})$ ("written without the separator bar") and $\mathbf{z}' = (\mathbf{x}', -1)$ is the $n+1$-vector with $\mathbf{x}$ first and element $n+1$ being -1. To have $\mathbf{Ax} - \mathbf{b} = \mathbf{0}$, we want $\mathbf{Mz} = \mathbf{0}$, and if $|\mathbf{M}| \neq \mathbf{0}$, then this is only possible for $\mathbf{z} = \mathbf{0}$. But $z_{n+1} = -1$!

If $|\mathbf{M}| = 0$: eliminate on to contradiction or square coeff. matrix.

Not an exam question by itself, but potentially useful for the exam – especially the last "But", where lots of errors are made:

Can determinants say anything if fewer eq's than var's?

Yes. If there are m equations, coeff. matrix $= \mathbf{A}$, and we can find a nonzero $m \times m$ determinant from m columns from $\mathbf{A}$, then:

- choose the *other* variables free; "move them over to the RHS"
- now we have an $m \times m$ with nonzero determinant, in the m variables corresponding to the column numbers.
- (We knew this already for the situation when we have row-echelon form ("staircase") – but it holds more generally.)

Theory note: If there are fewer eq's than var's, then there cannot be unique solution!

But: There does not have to exist any solution at all!

(Example: consider last example slide 5, throw in an extra fourth column $(0,\ 0,\ 0)'$ to make it a system in four variables – and there is still no solution if $c \neq 3$.)

Case: n eq's uniquely determining n variables as $\mathbf{A}^{-1}\mathbf{b}$.

- If you already know $\mathbf{A}^{-1}$, then calculating $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is easy.
- If you do not know $\mathbf{A}^{-1}$, and do not need anything but $\mathbf{A}^{-1}\mathbf{b}$, you (usually) save work by solving for $\mathbf{x}$ rather than inverting.
- But sometimes you want a *formula* (like " $\mathbf{A}^{-1}\mathbf{b}$ ") rather than a recipe ("Gaussian elimination").
- Soon: A formula for $\mathbf{A}^{-1}$.
- But first: *Cramér's rule* gives a formula for *each individual element* of a unique solution $\mathbf{x}$.
  - Usage: E.g., the macro model slide 2: what if you are only interested in output Y? Or only consumption C?
  - (Under the hood, you don't need to know details: Cramér's rule is just another guise of the formula for the inverse.)

$n \times n$, unique x cont'd: Cramér's rule

Consider $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is $n \times n$ and $|\mathbf{A}| \neq 0$.

- Form the determinant D_j by replacing column j of $\mathbf{A}$ by $\mathbf{b}$.

  and calculating the resulting determinant, goes without saying?

- Then $x_j = D_j / |\mathbf{A}|$.

Example (from slide 4):
Q1: What is D_3?
Q2: If $t = 4$, what is z?

$$\underbrace{\begin{pmatrix} 4 & 3 & 2 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{pmatrix}}_{=\mathbf{A}_t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}.$$

A1: Replace column 3 of $\mathbf{A}$ by the RHS and calculate the determinant $\begin{vmatrix} 4 & 3 & t \\ 1 & t & t \\ -2 & -3 & t \end{vmatrix} = t \cdot \left(\begin{vmatrix} 1 & t \\ -2 & -3 \end{vmatrix} - \begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 1 & t \end{vmatrix} \right)$

$= t \cdot (2t - 3 - (-12 + 6) + 4t - 3) = \underline{\underline{6t^2}}$

(For details on the cofactor expansion, see slide 27ff. in this version.)

Exercise: find the FLAWS in the arguments:

(I) "Let $\mathbf{A}$ be $n \times n$, and suppose that $\mathbf{b}$ is in fact one of the columns $\mathbf{A}$, namely number j. Then $D_j = |\mathbf{A}|$, and so x_j must be $= \frac{D_j}{|\mathbf{A}|} = 1$."

(II) "From the previous slide and slide 4, $D_3 = 6t^2 = \frac{-t}{2}|\mathbf{A}_t|$, so z must be $= \frac{-t}{2}$. Therefore, $\underline{\underline{z = -2}}$ when $t = 4$."

The flaw: Cramér's rule requires $|\mathbf{A}| \neq 0$! If that were verified, the arguments would be valid. ((II): OK iff $t \neq 0$, so OK for $t = 4$.)

Exercise: What can you actually say about x_j in case (I)? (Do you even know that a solution exists?)

Answer: there is a solution with $x_j = 1$ and all other $x_i = 0$ (check it!) – but if $|\mathbf{A}| = 0$, that solution is not unique; and, depending on $\mathbf{A}$, there *could* be solutions with $x_j \neq 1$ (dumb example: $\mathbf{0x} = \mathbf{0}$).

Back to the inverse: We have:

- Definition (and for square matrices, suffices to check ...)
- Exists iff $\mathbf{A}$ square AND $|\mathbf{A}| \neq 0$.
- How to find by Gaussian elimination

Next:

- A general formula based on cofactors
- Example(s)
- Rules! Old ones and some more.
- More examples?

... and then we should be done with linear algebra. Review topics, anyone? (Yes, as indicated on slide 1.)

A formula for the inverse:

Fix $\mathbf{A}$ (square). Let $\mathbf{C}$ have elements C_{ij}, where C_{ij} is the cofactor of element (i, j) of $\mathbf{A}$. Then $\mathbf{C}'\mathbf{A} = \mathbf{A}\mathbf{C}' = |\mathbf{A}|\,\mathbf{I}$, so:

Provided $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{C}'$ (beware: transpose!)

Terminology, not essential to remember: $\mathbf{C}' =: \texttt{adj}(\mathbf{A})$, abbreviation for "adjugate". Or "adjunct" / "classical adjoint".

"Workload" for $\mathbf{C}'$:
n^2 cofactors, each cofactor is
$\pm$ an $(n-1)\times(n-1)$ determinant, each determinant is
a sum of $(n-1)!$ terms, ...

Beauty is in the eye of the beholder? (clickable – won't compute the inverse of this, but ...)

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[not stressed in class] Just browsed through, and you won't be asked to reproduce this "proof" (which it really isn't, we have only postulated that cofactor expansion works). But see if you can *follow* the steps!

Again: If $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathtt{adj}\,(\mathbf{A})$ where $\mathtt{adj}(\mathbf{A}) := \mathbf{C}'$

Why? The following is an exercise in determinant rules:

- Recall cofactor expansion by ith row: $|\mathbf{A}| = \sum_{k=1}^{n} a_{ik}C_{ik}$
- If instead we picked the cofactors from another ("alien") row $\ell \neq i$, then we actually have $\sum_{k=1}^{n} a_{ik}C_{\ell k} = 0$.
  Why? Since cofactors of row ℓ do not depend on elements in row ℓ, then this is the determinant of the matrix we get by replacing row ℓ by a copy of row i. But that has two equal rows!
- To check $\mathbf{AC}' = |\mathbf{A}|\,\mathbf{I}$, check each element (i, j); it equals row $\#i$ from $\mathbf{A}$ dot row $\#j$ from $\mathbf{C}$ (because the prime!) i.e. $\sum_{k=1}^{n} a_{ik}C_{jk}$, which equals:

  $|\mathbf{A}|$ if $i = j$ (i.e. on the main diagonal), and 0 otherwise.
- Exercise: check element (i, j) of $\mathbf{C}'\mathbf{A}$. (Verifies that you only need to calculate "one of the products", if $\mathbf{A}$ is square.)

Example (from slide 4 with $t = 4$, again): To invert $\mathbf{A}_4 = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 4 & -1 \\ -2 & -3 & -4 \end{pmatrix}$, we calculate cofactors. (Remember signs!)
Then transpose, which is why the following cofactors appear "in the transposed order" (e.g. C_{21} on the $(1,2)$ position):

$$C_{11} = \begin{vmatrix} 4 & -1 \\ -3 & -4 \end{vmatrix} \qquad C_{21} = -\begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} \qquad C_{31} = \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix}$$

$$C_{12} = -\begin{vmatrix} 1 & -1 \\ -2 & -4 \end{vmatrix} \qquad C_{22} = \begin{vmatrix} 4 & 2 \\ -2 & -4 \end{vmatrix} \qquad C_{32} = -\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix}$$

$$C_{13} = \begin{vmatrix} 1 & 4 \\ -2 & -3 \end{vmatrix} \qquad C_{23} = -\begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix} \qquad C_{33} = \begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix}$$

So $$\begin{pmatrix} 4 & 3 & 2 \\ 1 & 4 & -1 \\ -2 & -3 & -4 \end{pmatrix}^{-1} = \frac{-1}{48} \begin{pmatrix} -19 & 6 & -11 \\ 6 & -12 & 6 \\ 5 & 6 & 13 \end{pmatrix}$$

How does this formula give the expression for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$?

Clarification of wording: "Find an expression for" means "valid as long as exists". This problem intended as example. [But actually: sometimes, even less is asked! See Messages & next week's lectures!]

On to work: Remembering to transpose, cofactors are as follows:

of the "a" element: d — of the "c" element: $-b$

of the "b" element: $-c$ — of the "d" element: a

(the "$-$"s for the cofactors of elements $(2, 1)$ and $(1, 2)$: "chessboard".)

And so the answer is the familiar(?) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Back to wording: If asked for "an expression for the inverse", this is the answer – because it is an expression valid as long as the inverse exists. (Writing "as long as $ad \neq bc$" not required, but won't hurt.)

Example: Use the formula to find *an expression for*

$$\begin{pmatrix} p_1 & 0 & 0 & p_2 \\ 0 & p_4 & p_3 & 0 \\ 0 & p_2 & p_1 & 0 \\ p_3 & 0 & 0 & p_4 \end{pmatrix}^{-1}$$

(16 cofactors, each size 3×3, *yikes!* But we are in some luck ...)

Symmetries! Let $K = \left|\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right|$. It is the determinant of the "middle 2×2 block" (check sign!) and also of "the corners put together".

"The cofactors of the corners" have a common factor K: We have $C_{11} = p_4 K$ and $C_{44} = p_1 K$. Beware signs: $C_{14} = -p_3 K$ and $C_{41} = -p_2 K$.

"The cofactors of the middle block" have a common factor K: We have $C_{22} = p_1 K$, $C_{33} = p_4 K$ and (beware signs) $C_{23} = -p_2 K$, $C_{32} = -p_3 K$.

"The cofactors of the greens are all zero!" Each has two proportional rows or two proportional columns. Example: $C_{12} = -\begin{vmatrix} 0 & p_4 & 0 \\ 0 & p_6 & 0 \\ p_7 & 0 & p_8 \end{vmatrix} = 0$. Remains:

- Put together the elements into $\mathbf{C}'$ (remember to transpose!)
- Calculate the determinant. We have all the cofactors.

cont'd: The determinant is (say!) $p_1 C_{11} + 0 + 0 + p_3 C_{41}$
(Q: why no "$-$" before p_3?) – which $= p_1 p_4 K - p_3 p_2 K = K^2$.

Stacking up cofactors, transposing:

$$\mathbf{C}' = \begin{pmatrix} p_4K & 0 & 0 & -p_2K \\ 0 & p_1K & -p_3K & 0 \\ 0 & -p_2K & p_4K & 0 \\ -p_3K & 0 & 0 & p_1K \end{pmatrix}$$

and the expression for the inverse is:

$$\frac{1}{K}\begin{pmatrix} p_4 & 0 & 0 & -p_2 \\ 0 & p_1 & -p_3 & 0 \\ 0 & -p_2 & p_4 & 0 \\ -p_3 & 0 & 0 & p_1 \end{pmatrix}$$

Exercise: verify! (Multiply and see that you get the identity.)

Terminology: "invertible" $\leftrightarrow$ has an inverse.

But also: A square matrix is called *non-singular* if it has an inverse, and *singular* if it does *not.*

("Singular" is fairly common. You can use "non-invertible" on the exam.)

Rules for the inverse. Known already:

- If $\mathbf{A}$ is square and furthermore either $\mathbf{AM} = \mathbf{I}$ or $\mathbf{MA} = \mathbf{I}$, then $\mathbf{A}^{-1}$ exists and equals $\mathbf{M}$.
- ... and if so: $\mathbf{M}^{-1}$ exists and equals $\mathbf{A}$. Consequently:
- If $\mathbf{A}$ is invertible, then so is $\mathbf{A}^{-1}$, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
- $\mathbf{A}$ is invertible $\Leftrightarrow$ square and $|\mathbf{A}| \neq 0$.
- If so, we have the formula – and the Gaussian elm'n method.

A few more rules next slide

More rules for the inverse. (The term paper problem set stops way before this!)

- Iff exists: $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$. (Because $1 = |\mathbf{I}| = |\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{A}|\ |\mathbf{A}^{-1}|$)
- Either $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$ or neither inverse exists.
- Let $k \in \mathbb{N}$. Then $\mathbf{A}^k$ is invertible iff $\mathbf{A}$ is, and if so: $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$. Which we denote $\mathbf{A}^{-k}$.
  - Easy to check. Determinant $\neq 0$ and $\mathbf{A} \ldots \mathbf{A}\mathbf{A}^{-1} \ldots \mathbf{A}^{-1} = \mathbf{I}$
  - (Often seen, not not exam relevant: $\mathbf{A}^0 = \mathbf{I}$, cf. $\alpha^0 = 1$.)
- If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then either $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ or neither side exists. Note reverse order!
  - Similarly, $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$ or neither side exists. And also: $(c\mathbf{A})^{-1} = \mathbf{A}^{-1}(c\mathbf{I})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$ if $c \neq 0$.
  - "Proof": Easy to check if well-defined: $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$.
  - Beware: $(\mathbf{AB})^{-1}$ could exist even if $\mathbf{A}$ and $\mathbf{B}'$ are $m \times n$ with $m \neq n$.
    - Or more factors, e.g. $\mathbf{x}'\mathbf{A}\mathbf{x}$ is 1×1 and has an inverse iff $\neq 0$
    - (Not curriculum: That's why you in e.g. econometrics may see "large" products $(\ldots)^{-1}$ not written out. BTW, it requires $m < n$, impossible if $m > n$.)

Example cases based on some book problems. *[not stressed in class]*

- Let $|\mathbf{X}'\mathbf{X}| \neq 0$. Show that $\mathbf{A} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ satisfies $\mathbf{A} = \mathbf{A}^2$. What can we say about $|\mathbf{A}|$?
    - We have not assumed $\mathbf{X}$ square, so we cannot simplify $\mathbf{A}$. But $\mathbf{A}^2 = \mathbf{I} - 2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ which $= \mathbf{I} + (-2+1)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ because $\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{I}$.
    - $|\mathbf{A}| = |\mathbf{A}^2|$, so in Math 2 we can tell that $|\mathbf{A}|$ is zero or one.
      The proof that it is 0 uses only Math2, but is too tricky for a Math2 exam: Suppose *for contradiction* that $\mathbf{A}^{-1}$ exists: Then $\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A}$, and $\mathbf{A} = \mathbf{I}$. And so $\mathbf{0} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Right-multiply by $\mathbf{X}$ to get: $\mathbf{0} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X} = \mathbf{X}\mathbf{I}$. But $\mathbf{X} = \mathbf{0} \Rightarrow |\mathbf{X}'\mathbf{X}| = 0$, contradiction.
- Suppose some power $\mathbf{B}^k$ is $\mathbf{0}$. Then $\mathbf{B}^{-1}$ does not exist, but $(\mathbf{I} - \mathbf{B})^{-1}$ does and equals $\mathbf{I} + \mathbf{B} + \ldots + \mathbf{B}^{k-1}$. (Cf. geometric series.)
    - $|\mathbf{B}^k| = |\mathbf{B}|^k$ and also $= |\mathbf{0}|$.
    - Calculate $(\mathbf{I} + \mathbf{B} + \ldots + \mathbf{B}^{k-1})(\mathbf{I} - \mathbf{B}) = [\ldots] = \mathbf{I} - \mathbf{B}^k = \mathbf{I}$.
- Suppose $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then $\mathbf{M}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$, all $k \in \mathbb{N}$.

  Show: Valid even for negative integers k iff $|\mathbf{D}| \neq 0$.
  Typically want: $\mathbf{D}$ diagonal. *Explain:* Why is this convenient?
  (Computes $\mathbf{M}^k$ when k large or ... sometimes even non-integer. Curriculum at BI Norwegian Business School and NTNU øk.ad. Application: $k = 1/2$ to standardize a random vector with covariance matrix $\mathbf{M}$.)

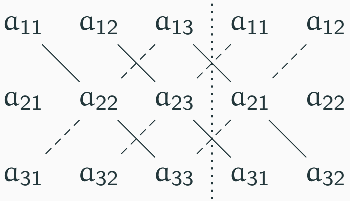
Two applications in this course *[not stressed in class]*

Fix a function $f \in C^2$ of n variables: $f(\mathbf{x})$. Let $\mathbf{H}$ be the so-called Hessian matrix: $h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ (symmetric matrix, depends on $\mathbf{x}$)

- *Second-order approximation around $\mathbf{x}_*$:* the 2nd-order term will become $\frac{1}{2}(\mathbf{x} - \mathbf{x}_*)'\mathbf{H}_*(\mathbf{x} - \mathbf{x}_*)$ where $\mathbf{H}_*$ indicates that it is evaluated at $\mathbf{x}_*$. Prime denotes transpose; if you don't like that (with derivatives in the picture): $\frac{1}{2}(\mathbf{x} - \mathbf{x}_*) \cdot (\mathbf{H}_*(\mathbf{x} - \mathbf{x}_*))$. Note $\mathbf{H}_*()$ means product, not "of".
- Behind the scenes, this underlies the *2nd derivatives test* in n variables, and *concavity/convexity tests*. Case $n = 2$ in Math2:
  - The *Hessian determinant* $|\mathbf{H}|$ equals the "$AC - B^2$" from your first Math course's 2nd derivative test. (Math2 does not require you to use *matrix formulation* in your 2nd derivative tests, but the content is the same anyway!)
  - If both the $|\mathbf{H}|$ and the top-left element A are > 0:
    - everywhere, the function is convex
    - merely at some stationary point, then this is strict local min.
  - (Why do we have opposite signs $|\mathbf{H}| > 0 > A$ for concavity/max? Switch sign on f and thus on A; but since $n = 2$, then $|-\mathbf{H}| = (-1)^2|\mathbf{H}|$, no sign change! More than 2 variables: Math3!)

By popular demand (I): The Sarrus rule for calculating 3×3 determinants. *CAVEAT: NOT VALID for larger!*

Look at the picture:



To the left, matrix elements. To the right, the first two columns repeated.

$$\text{Determinant} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -\left[a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}\right] \end{array}$$

- The blue ones (first line) are the triplets connected with lines Northwest–Southeast.
- The red ones – which get subtracted, 2nd line – are the triplets connected with dashes Northeast–Southwest.

Example: $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 7 & 0 & -5 \end{vmatrix}$. Becomes $\begin{matrix} 1 & 2 & 4 & \vdots & 1 & 2 \\ -1 & 8 & 3 & \vdots & -1 & 8 \\ 7 & 0 & -5 & \vdots & 7 & 0 \end{matrix}$

when we write the elements and then repeat the first two columns.

The ones left as-is: start top-left, go south-east:

$$\begin{matrix} 1 & 2 & 4 & \vdots & & \\ & 8 & 3 & \vdots & -1 & \\ & & -5 & \vdots & 7 & 0 \end{matrix}$$

cyan product:	$1 \cdot 8 \cdot (-5)$	$=$	-40
green product:	$2 \cdot 3 \cdot 7$	$=$	42
b/w product:	$4 \cdot (-1) \cdot 0$	$=$	0

The ones to subtract/change sign are the south-*west* connections:

$$\begin{matrix} & & 4 & \vdots & 1 & 2 \\ & 8 & 3 & \vdots & -1 & \\ 7 & 0 & -5 & \vdots & & \end{matrix}$$

magenta product:	$4 \cdot 8 \cdot 7$	$=$	224
b/w product:	$1 \cdot 3 \cdot 0$	$=$	0
yellow product:	$2 \cdot (-1) \cdot (-5)$	$=$	10

Determinant $= -40 + 42 + 0 - \left[224 + 10\right] = -232$.

By popular demand (II): More cofactor expansion

Recall expansion along the ith row:

- Pick one row number i (and stick to it!)
- For each element in the row, multiply it by its cofactor
  (...remember what a "cofactor" is, and in particular: the chessboard for signs)
- Add up.

Or pick a column instead of a row.

3×3 example from Tuesday's class: $\begin{vmatrix} 4 & 3 & p \\ 1 & t & q \\ -2 & -3 & r \end{vmatrix}$ by 3rd column.

Cofactor of element $(1,3)$: strike out first column and third row, evaluate determinant of rest, (chessboard says: do not switch sign)

$\begin{vmatrix} \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ 1 & t & \text{[redacted]} \\ -2 & -3 & \text{[redacted]} \end{vmatrix}$. That is, the cofactor is $-3 + 2t$. (cont'd) example cont'd:

Have already: the cofactor of the "p" element $(1,3)$ is $2t-3$.

The cofactor of the "q" element $(2,3)$ is $-\begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix} = 6$; the negative sign due to the chessboard.

The cofactor of the "r" element $(3,3)$ is $\begin{vmatrix} 4 & 3 \\ 1 & t \end{vmatrix} = 4t-3$.

Multiply each by the element and add up:
$p(2t-3) + 6q + r(4t-3) = 2t(p+2r) - 3(p-2q+r)$.

- For $|\mathbf{A}_t|$ on slide 4, $p = 2,\ q = -1,\ r = -4$ and we get $-12t$.
- For the D_3 on slide 12, $p = q = r = t$, so we get $6t^2$.
- As long as $t \neq 0$ so the $\mathbf{A}_t$ slide 4 has an inverse: Cramér's rule on the equation system $\mathbf{A}_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$, says that $z = \frac{2t(p+2r)-3(p-2q+r)}{-12t}$.

3rd variable "z" because we replaced 3rd column.

4×4 **example:** $\begin{vmatrix} 1 & 2 & 4 & -7 \\ -1 & 8 & 3 & 3 \\ 7 & 0 & -5 & 1 \\ 2 & 2 & -3 & 5 \end{vmatrix}$ in a boring way, no "clever" shortcuts.

First: pick one row or column. I pick column 4. Need the four cofactors (top to bottom, none involving column 4):

$$C_{14} = (-1)^5 \begin{vmatrix} -1 & 8 & 3 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix} \qquad C_{24} = (-1)^6 \begin{vmatrix} 1 & 2 & 4 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix}$$

$$C_{34} = (-1)^7 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 2 & 2 & -3 \end{vmatrix} \qquad C_{44} = (-1)^8 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 7 & 0 & -5 \end{vmatrix}$$

The determinant will be $-7 \cdot C_{14} + 3 \cdot C_{24} + 1 \cdot C_{34} + 5 \cdot C_{44}$.

But the cofactors involve 3×3 determinants that must be calculated. For example by cofactor expansion.

4×4 example cont'd: since this is a cofactor expansion exercise, use that method for every 3×3 too. Arbitrary choices: first row for the first two, second column for the last two.

C_{14} by first row: $(-1)^5 \begin{vmatrix} -1 & 8 & 3 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix} =$

$(-1)^5 \cdot \Big\{ -1 \cdot \text{its cofactor} + 8 \cdot \text{its cofactor} + 3 \cdot \text{its cofactor} \Big\}$, all cofactors relative to that 3×3 determinant (ignore for the moment that there was ever a 4×4).

$$(-1)^5 \cdot \Big\{ -1 \cdot \underbrace{(-1)^{1+1} \begin{vmatrix} 0 & -5 \\ 2 & -3 \end{vmatrix}}_{=0-(-10)} + 8 \cdot \underbrace{(-1)^{1+2} \begin{vmatrix} 7 & -5 \\ 2 & -3 \end{vmatrix}}_{=-(-21+10)} + 3 \underbrace{(-1)^{1+3} \begin{vmatrix} 7 & 0 \\ 2 & 2 \end{vmatrix}}_{=14-0} \Big\}$$

which equals $10 - 8 \cdot 11 - 3 \cdot 14 = -120$, and so the "$-7 \cdot C_{14}$" contribution is $-7 \cdot (-120) = 840$. On to the three others.

4×4 example cont'd: $C_{24} = (-1)^6 \begin{vmatrix} 1 & 2 & 4 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix}$ by first row.

$$1 \cdot (-1)^2 \left| \text{[what?]} \right| + 2 \cdot (-1)^3 \left| \text{[what?]} \right| + 4 \cdot (-1)^4 \left| \text{[what?]} \right|$$

(Go ahead, fill in!)

$$1 \cdot (-1)^2 \underbrace{\begin{vmatrix} 0 & -5 \\ 2 & -3 \end{vmatrix}}_{=10} + 2 \cdot (-1)^3 \underbrace{\begin{vmatrix} 7 & -5 \\ 2 & -3 \end{vmatrix}}_{=-11} + 4 \cdot (-1)^4 \underbrace{\begin{vmatrix} 7 & 0 \\ 2 & 2 \end{vmatrix}}_{=14}$$

which sums up to $10 + 22 + 56 = 88$.

So the "$3 \cdot C_{24}$" contribution is 264. Then what next?

4×4 example cont'd (this slide from class):

$$C_{34} = (-1)^7 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 2 & 2 & -3 \end{vmatrix} \text{ by second column:}$$

(The first "$-$" is the $(-1)^7$. The negative signs before the "2" are from the "chessboard of signs" for the 3×3):

$$-\left\{ -2 \begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} + 8 \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} \right\}$$

$$= -\left\{ -2(3-6) + 8(-3-8) - 2(3+4) \right\} = 96.$$

Contribution to determinant: $1 \cdot 96$. Then finally the contribution $5 \cdot C_{44}$: C_{44} is the same determinant as in the Sarrus example: -232. (Time ran out in class; finding it by expanding along the 2nd column $\rightsquigarrow$ exercise!) So the 4×4 determinant is $840 + 264 + 96 + 5 \cdot (-232) = 40$.

(As one of you asked: once you have reduced it to 3×3's, you are free to use Sarrus on those! Or row/column operations ...)