Lecture Notes 1-2

Ola
kjempefintola@hotmail.com

Exponential functions

$f(x) = A\underset{\hookrightarrow \text{ base}}{a}^{x \to \text{ exponent}}$, $a>0$.

Let the variable x be time, then

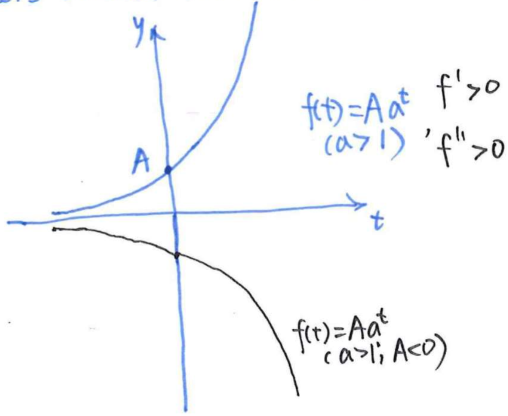
$f(t) = Aa^t$.

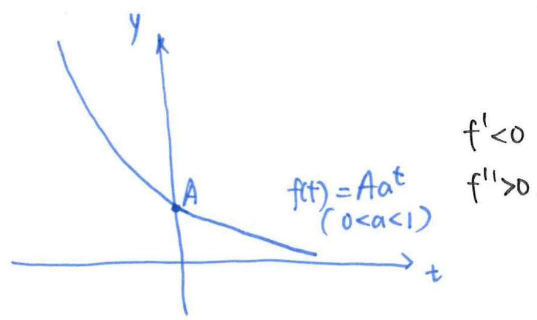
- $f(t+1) = Aa^{t+1} = \underbrace{Aa^t}_{f(t)} \cdot a = af(t)$ for all t. $\Rightarrow$ $f(t+1) = af(t)$, $\forall t$.

f $\underbrace{\text{increases}}_{a>1}$ (or $\underbrace{\text{decreases}}_{0<a<1}$) by a fixed factor a per unit of time. $\Rightarrow$ increase (or decrease) exponentially.

- $f(0) = A \cdot a^0 = A$. we can also write $f(t) = f(0) \cdot a^t$.

Let's consider $A>0$, $a>0$.





The Natural Exponential Function.

$f(x) = a^x$.

Each base a gives different exponential function.

One perticalar base number is far more important than others

$f(x) = e^x$. $e = 2.718281828459045 \ldots$

$\hookrightarrow$ natural exponential function.

Rules: a) $e^s e^t = e^{s+t}$ b). $e^s / e^t = e^{s-t}$. c) $(e^s)^t = e^{st}$.

sometimes the notation $\exp(x)$ or even $\exp x$. are used

for example $\exp(x^3 + x\sqrt{x+1/x} + 5)$ is easier to write and read than

$e^{x^3 + x\sqrt{x+1/x} + 5}$

Logarithmic functions

$a>0$.

$y = a^x$. if we know the exponent x, we can get $y = a^x$.

if we know y, and want to calculate the exponent x, $x = \log_a y$

~~z=?~~ $4 = 2^x$. $9 = 3^x$.

$y = \log_a x$ y is called the logarithm of x to base a.

$\log_a x$ is the power of a you need to get x. Ex: $\log_2 4 = 2$

$\log_3 27 = 3$

Also: $a^y = x \Rightarrow a^{\log_a x} = x$.

$b^{\log_b x} = x$,

Some Rules:

a). $\log_a(xy) = \log_a x + \log_a y$

Proof: Let $z_1 = \log_a x$, $z_2 = \log_a y \Rightarrow a^{z_1} = x$, $a^{z_2} = y$.

$\Rightarrow a^{z_1+z_2} = xy \Rightarrow \log_a xy = z_1 + z_2 = \log_a x + \log_a y$.

b). $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

c) $\log_a x^p = p\log_a x$, $x>0$.

Proof: Let $z = \log_a x^p \Rightarrow a^z = x^p$, $\Rightarrow$ $\cancel{(a^z)^{\frac{1}{p}}}$ $(a^z)^{\frac{1}{p}} = (x^p)^{\frac{1}{p}}$

$\Rightarrow a^{\frac{z}{p}} = x$. $\Rightarrow \frac{z}{p} = \log_a x$. $\Rightarrow z = p\log_a x \Rightarrow \log_a x^p = p\log_a x$.

d) $\log_a 1 = 0$ and $\log_a a = 1$.

Another way to prove Rule a).

$a^{\underline{\log_a(xy)}} = xy = (a^{\log_a x})(a^{\log_a y}) = a^{\underline{\log_a x + \log_a y}} \Rightarrow \log_a(xy) = \log_a x + \log_a y$

Natural logarithm

$y = \log_a x \xrightarrow{\text{take "}e\text{" as base}} y = \log_e x$, writen as $\underline{y = \ln x}$.

$e^{\ln x} = x$.

Rules a) – d) apply.

$\ln(xy) = \ln x + \ln y$; $\ln(\frac{x}{y}) = \ln x - \ln y$; $\ln x^p = p \ln x$; $\ln 1 = 0$; $\ln e = 1$

Also, sometimes it's useful to know:

$$\underline{\log_a b = \frac{\ln b}{\ln a}}, \quad a \neq 1.$$

Proof: Let $z = \log_a b \Rightarrow a^z = b \Rightarrow \ln a^z = \ln b \Rightarrow z \ln a = \ln b$

$\Rightarrow z = \frac{\ln b}{\ln a} \Rightarrow \log_a b = \frac{\ln b}{\ln a}$.

$\cancel{\log_a b = \frac{\log_c b}{\log_c a}}$.

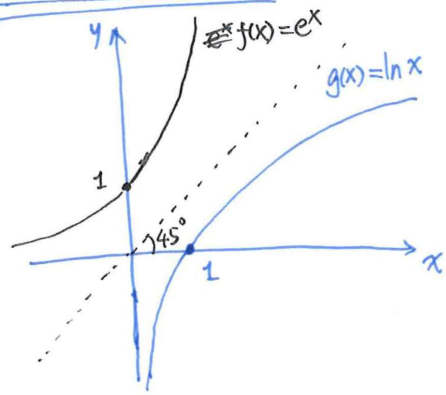
Ex: $e^x + 4e^{-x} = 4$, $x = ?$ $\quad e^x = \dots \Rightarrow x = \ln \dots$

$e^x + \frac{4}{e^x} = 4 \Rightarrow (e^x)^2 + 4 = 4e^x \Rightarrow (e^x)^2 - 4e^x + 4 = 0$

$\Rightarrow (e^x - 2)^2 = 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2$.

you could also let $u = e^x$ first.

Function $g(x) = \ln x$



domain $x > 0$.

$\lim_{x \to 0^+} \ln x \to -\infty$

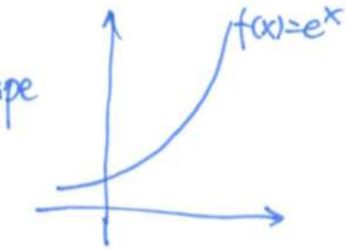
$\lim_{x \to \infty} \ln x \to +\infty$

Derivative of the Natural ~~tog~~exponential functions

$f(x) = e^x \Rightarrow f'(x) = e^x$.

Also $f''(x) = e^x$.

$\left.\begin{array}{l} \cancel{f'>0}\ f'(x) > 0 \Rightarrow \text{increasing} \\ f''(x) > 0 \Rightarrow \text{convex}. \end{array}\right\}$ confirming the shape



Differentiating other exponential functions.

$g(x) = y = a^x, \quad a > 0$.

$a = e^{\ln a} \Rightarrow y = a^x = (e^{\ln a})^x = e^{x \cdot \ln a}$

$g'(x) = y' = e^{x \cdot \ln a} \cdot \ln a = \cancel{a}\, a^x \cdot \ln a$

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$\underline{\underline{(a^x)' = a^x \cdot \ln a}}$.

Ex: $y = x^5 e^x$

$y' = \underline{5 \cdot x^4 \cdot e^x + x^5 e^x} = \cancel{5x}\ x^4 e^x (5 + x)$

$y'' = 20x^3 e^x + 5x^4 e^x + 5x^4 e^x + x^5 e^x$

$= x^3 e^x (20 + 10x + x^2)$

Ex: $y = \frac{e^x}{x} = e^x \cdot x^{-1}$.

$y' = e^x \cdot x^{-1} + e^x \cdot (-1) \cdot x^{-2} = e^x \cdot x^{-1} - e^x \cdot x^{-2} = \frac{e^x \cdot (x-1)}{x^2}$

$y'' = e^x \cdot x^{-1} + (-1) \cdot e^x \cdot x^{-2} - [e^x \cdot x^{-2} + e^x \cdot (-2) \cdot x^{-3}]$

$= e^x \cdot x^{-1} - 2e^x \cdot x^{-2} + 2e^x \cdot x^{-3}$

Ex: $f(x) = 10^{-x}$. $\quad \underline{(a^x)' = a^x \ln a}$

$f'(x) = 10^{-x} \ln 10 \cdot (-1) = -10^{-x} \ln 10$.

Ex: $g(x) = x \cdot 2^{3x}$.

$\frac{d}{dx} 2^{3x} = 2^{3x} \ln 2 \cdot 3 = 3 \cdot 2^{3x} \ln 2$

$g'(x) = 2^{3x} + x \cdot (3 \cdot 2^{3x} \cdot \ln 2)$

$= 2^{3x}(1 + 3x \ln 2)$

<u>Derivative of Natural logarithmic functions</u>

$g(x) = \ln x$.

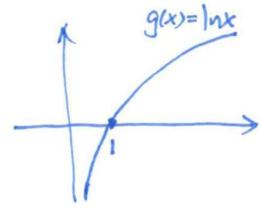
$g'(x) = \frac{1}{x}$.

Proof: $e^{g(x)} = e^{\ln x} = x$

$\Rightarrow e^{g(x)} = x \xrightarrow{\text{differentiate w.r.t } x} \underbrace{e^{g(x)}}_{=x} \cdot g'(x) = 1$

$\Rightarrow x \cdot g'(x) = 1 \Rightarrow g'(x) = \frac{1}{x}$.

Recall $x > 0$. $\Rightarrow \left\{ \begin{array}{l} g'(x) = \frac{1}{x} > 0 \Rightarrow \text{strictly increasing} \\ g''(x) = -\frac{1}{x^2} < 0 \Rightarrow \text{strictly concave "}\frown\text{"} \end{array} \right\}$



- $y = \ln(h(x))$ $\quad h(x)$ differentiable and positive.

$\Rightarrow y' = \frac{1}{h(x)} h'(x) = \frac{h'(x)}{h(x)}$ = rate of change in $h(x)$ per unit of change in x.

For example. Let $N(t)$ be population, which changes over time.

Then $\frac{d}{dt} \ln(N(t)) = \frac{N'(t)}{N(t)} \Rightarrow$ growth rate of $N(t)$

Ex: $y = \ln(4-x^2)$

Domain: $4 - x^2 > 0 \Rightarrow x^2 < 4 \Rightarrow -2 < x < 2$

$$y' = \frac{1}{4-x^2} \cdot (-2x) = \frac{2x}{x^2-4}$$

When differentiating an expression containing products, quotients, powers, and combinations of these, it is often easier to use logarithmic differentiation. $\cancel{x^2 \cdot 3x}$

Ex: $y = x^x \quad y' = ?$

$$\underline{\ln y} = \ln x^x = \underline{x \ln x} \xrightarrow{\text{diff. w.r.t. } x} \frac{y'}{y} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\Rightarrow y' = y(\ln x + 1) = \underline{x^x(\ln x + 1)}$$

Ex: $y = [A(x)]^\alpha \cdot [B(x)]^\beta \cdot [C(x)]^\gamma$

$$\ln y = \alpha \ln(A(x)) + \beta \ln(B(x)) + \gamma \ln(C(x))$$

$$\frac{y'}{y} = \alpha \frac{A'(x)}{A(x)} + \beta \frac{B'(x)}{B(x)} + \gamma \frac{C'(x)}{C(x)}.$$

$$y' = \ldots$$

Derivative of ~~Normal log~~. General log.

$$g(x) = \log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x.$$

$$g'(x) = \frac{1}{\ln a} \cdot \frac{1}{x}.$$

Interest rate

If the interest rate is 2% per year, then the two year effective rate is > 4%.

$$(1.02)^2 = [1 + 0.02]^2$$

$$= 1^2 + \underbrace{2 \cdot 0.02}_{\substack{=0.04 \\ \text{the "4\%"}}} + \underbrace{0.02^2}_{\text{interest on interest.}}$$

<u>Naive definition</u>

Expression: $\lim_{x \to a} f(x) = A$

means that we can make $f(x)$ as close to A as we want to for all x sufficiently close to, but not equal to, a.

<u>Rules</u>: $\lim_{x \to a} f(x) = A$ $\lim_{x \to a} g(x) = B$

(R1) $\lim_{x \to a} [f(x) \pm g(x)] = A \pm B$

works also if $A + B = \pm\infty$, but one if

$\begin{cases} A = +\infty \\ B = -\infty \end{cases}$ or $\begin{cases} A = -\infty \\ B = +\infty \end{cases}$

(R2) $\lim_{x \to a} [f(x) \cdot g(x)] = A \cdot B$

works also if for example $\begin{cases} A = +\infty \\ B \neq 0. \end{cases}$

but not if $A \cdot B$ takes the form of "$0 \cdot \infty$"

(R3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}$, if $B \neq 0$.

$\frac{A}{\pm\infty} = 0$ if $A \in \mathbb{R}$. but watch out for "$\frac{\pm\infty}{\pm\infty}$", "$\frac{0}{0}$"

(R4) $\lim_{x \to a} [f(x)]^r = A^r$. if A^r is defined and r is a real number.

Ex: $\lim_{x \to -2} (x^2 + 5x) = \cancel{\lim_{x \to -2} (x \cdot x + 5 \cdot x)}$

$= 4 - 10 = \cancel{8}\ -6.$

$\lim_{x \to 4} \frac{2x^{3/2} - \sqrt{x}}{x^2 - 15} = \frac{2 \cdot 8 - 2}{16 - 15} = 14.$

$\cancel{\lim}$