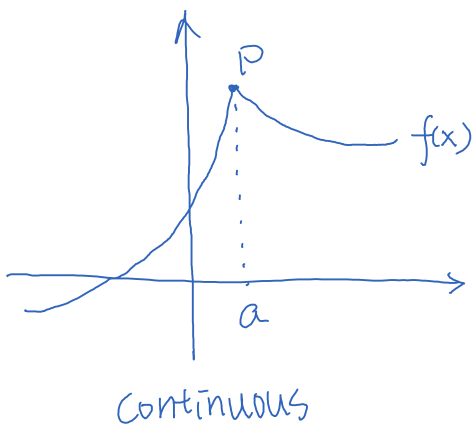
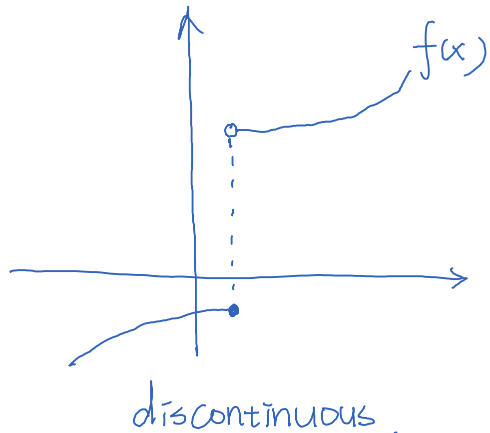
Lecture Notes 3-4

Continuity

Geometrically, a function is continuous on an interval if its graph is connected - that is, if it has no breaks , or "jumps".





Why do we care about continuity?
Numerical approximations
approximate $f(\sqrt{2})$ *by* $f(1.4142)$
Relies on continuity of f: *since* 1.4142 *is close to* $\sqrt{2}$, $f(1.4142)$ *must also be close to* $f(\sqrt{2})$
Continuity of functions usually reflect continuity of the represented phenomenon, in the sense of gradual rather than sudden changes.
Calculating limits

Definition

The function of f is continuous at $x=a$ iff $\lim_{x\to a} f(x) = f(a)$

i) f must be defined at $x=a$

ii) The limit of $f(x)$ as x tends to a must exist.

iii) This limit must be equal to $f(a)$

Also $f(a) = f(\lim_{x\to a} a)$

Continuity means that we can put "$\lim_{x\to a}$" inside the function.

Nearly all functions of this course are continuous everywhere where they are defined.

Ex: $f(x) = x^{-3}$ is continuous except at $x=0$, where it is not defined.

Properties

If f and g are continuous at a, then:

(a) $f+g$ and $f-g$ are continuous at a.

(b) fg and, in case $g(a) \neq 0$, the quotient f/g are continuous at a

(c). $[f(x)]^r$ is continuous at a, if $[f(a)]^r$ is defined, where r is a real number.

(d). If f has an inverse on the interval I, then its inverse f^{-1} is continuous on $f(I)$

Any function that can be constructed from continuous functions by combining one or more operations of addition, subtraction, multiplication, division (except by zero), and composition is continuous at all points where it is defined.

Ex: $f(x) = \dfrac{x^4+3x^2-1}{(x-1)(x+2)}$. continuous except at $\begin{cases} x=1 \\ x=-2 \end{cases}$

$g(x) = (x^2+2)\left(x^3+\frac{1}{x}\right)^4 + \dfrac{1}{\sqrt{x+1}}$

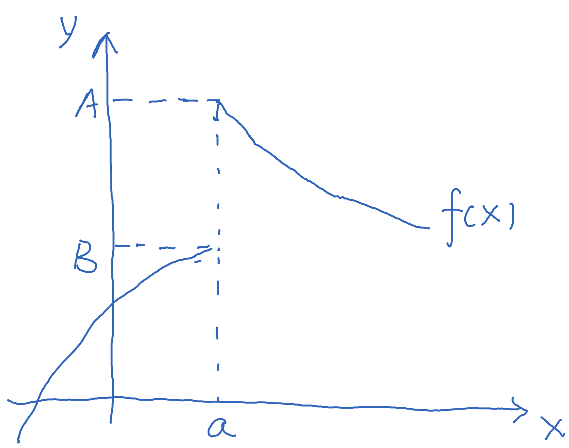
Domain: $x > -1$ and $x \neq 0$.

continuous in $(-1, 0) \cup (0, \infty)$

If f is continuous at $x = a$, then the limit of $f(x)$ as x tends to a is found simply by evaluating $f(a)$

Ex: $$\lim_{x \to 3} \frac{1}{1 + (\ln(1+e^{-x}))} = \frac{1}{1 + (\ln(1+e^{-3}))}$$

One-sided limits



Left limit:

$\lim_{x \to a^-} f(x) = B$ or $f(x) \to B$ as $\underline{x \to a^-}$.

tends to a from below.

Right limit:

$\lim_{x \to a^+} f(x) = A$ or $f(x) \to A$ as $\underline{x \to a^+}$

tends to a from above.

$$\lim_{x \to a} f(x) = A \iff \left[\lim_{x \to a^-} f(x) = A \text{ and } \lim_{x \to a^+} f(x) = A \right].$$

L'Hôpital's Rule

Ex:

L'Hôpital's Rule

Ex:
$$\lim_{x\to 0} \frac{x}{e^x-1} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 0} \frac{1}{e^x} = 1.$$

$\Downarrow$ differentiating both the numerator and denominator w.r.t. x.

L'Hôpital's Rule

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$

L could be finite, ∞, or $-\infty$.

Extensions:

“a” can be replaced by “a^-”, “a^+”, “$-\infty$” or “∞”

“$\frac{0}{0}$” can be replaced by “$\frac{\pm\infty}{\pm\infty}$”

Ex:
$$\lim_{x\to\infty} \frac{x^{1000}}{(1.0001)^x} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to\infty} \frac{1000x^{999}}{(1.0001)^x \ln(1.0001)} = \text{“}\frac{\infty}{\infty}\text{”}$$
$$= \lim_{x\to\infty} \frac{1000\cdot 999\cdot x^{998}}{(1.0001)^x(\ln(1.0001))^2} = \text{“}\frac{\infty}{\infty}\text{”} = \cdots = \lim_{x\to\infty} \frac{1000!}{(1.0001)^x(\ln(1.0001))^{1000}}$$
$$= 0$$

Note:

- We can use L'Hôpital's rule repeatedly as long as “$\frac{0}{0}$” “$\frac{\pm\infty}{\pm\infty}$”.
- Exponentials overwhelm powers

You must check that you really have "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

You must on the exam make the claim that you checked, by writing down the "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" $\quad \frac{0}{\frac{1}{\infty}} \quad \frac{\infty}{\frac{1}{0}}$

"$0 \cdot \infty$" can be rewritten as either "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

$$f(x)\cdot g(x) = \frac{f(x)}{1/g(x)} = \frac{g(x)}{1/f(x)}$$

Rewrite wisely!

Ex: $\lim\limits_{x\to 0^+} x\ln x \Rightarrow$ "0"$\cdot$"$-\infty$"

either $= \lim\limits_{x\to 0^+} \frac{\ln x}{x^{-1}} = \frac{"-\infty"}{\infty} = \lim\limits_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim\limits_{x\to 0^+}(-x) = 0$

or $= \lim\limits_{x\to 0^+} \frac{x}{(\ln x)^{-1}} = \text{"}\frac{0}{0}\text{"} = \lim\limits_{x\to 0^+} \frac{1\cdot x}{-(\ln x)^{-2}}$ gets even more complex.

$= \dots \lim\limits_{x\to 0^+} \frac{x}{(-1)^n(\ln x)^{-n}}$

Other ~~inter~~ indeterminate forms

"1^∞", "∞^0", "0^0"

solution: Find $\lim\limits_{x\to a} \ln(\ ) = y$. then e^y is your answer.

$\lim[\ln(1^\infty)] = \lim[\infty \ln 1] =$ "$\infty\cdot 0$" $=$ "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$" $= y. \Rightarrow e^y$.

Ex: $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$

$$\lim_{x\to\infty} \ln x^{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim_{x\to\infty}\ln x^{\frac{1}{x}}} = e^0 = 1.$$

$x = e^{\ln x}$.

"$\infty_1 - \infty_2$" $\sim \infty_1\left(1 - \text{"}\frac{\infty_2}{\infty_1}\text{"}\right)$

Ex: $\lim\limits_{x\to\infty} \frac{x^p}{a^x} = ?$ $\quad a>1$ and $p>0$.

$$\lim_{x\to\infty} \ln\frac{x^p}{a^x} = \lim_{x\to\infty} (\underbrace{p\ln x}_{\infty} - \underbrace{x\ln a}_{\infty}) = \lim_{x\to\infty}\left[\underbrace{\ln x}_{\infty}\left(p - \ln a \underbrace{\frac{x}{\ln x}}_{\text{"}\frac{\infty}{\infty}\text{"}}\right)\right]$$

$$\lim_{x\to\infty} \frac{x}{\ln x} = \text{``}\frac{\infty}{\infty}\text{''} = \lim_{x\to\infty} x = \infty$$

$$\lim_{x\to\infty} \left(p - \ln a \frac{x}{\ln x}\right) = -\infty$$

$$\lim_{x\to\infty} \left[\ln x \left(p - \ln a \frac{x}{\ln x}\right)\right] = -\infty$$

$$\lim_{x\to\infty} \frac{x^p}{a^x} = e^{-\infty} = \frac{1}{e^{\infty}} = 0.$$

Partial proof of L'Hôpital's rule

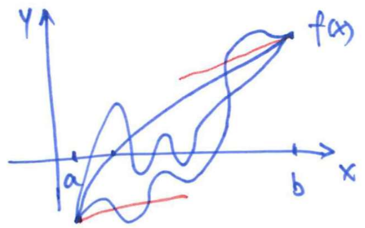
Provided $f(a) = g(a) = 0$.

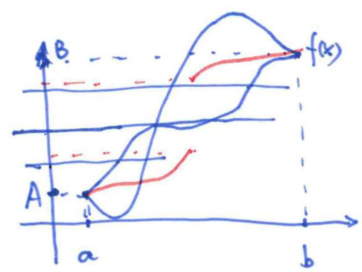
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \text{``}\frac{0}{0}\text{''}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)} = \frac{f'(a)}{g'(a)} = \cdots = L \Rightarrow \text{must be finite or } \pm\infty$$

$$\frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Intermediate value theorem





f is continuous on $[a, b]$.

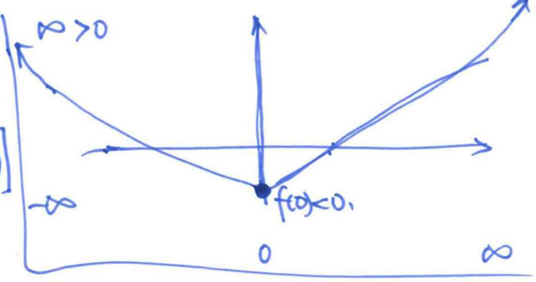
i). if $f(a)$, $f(b)$ have different signs, ~~there th~~ then there is at least one c in $[a, b]$ such that $f(c) = 0$.

ii) if $f(a) \neq f(b)$, then for every intermediate value y in the open interval between $f(a)$ and $f(b)$, there is at least one c in (a, b) such that $f(c) = y$.

Ex: show that $-x+e^x=e$ has at least two solutions. hints: look at $x=0$.

$f(x)=-x+e^x-e$. want to show $f(x)<0$ has at least two solutions.

$f(0)=e^0-e=1-e<0$.



$$\lim_{x\to\infty} f(x) = \text{"}-\infty\text{"} + \text{"}\infty\text{"}$$
$$= \lim_{x\to\infty}\Big[e^x\big(\underbrace{\underbrace{-\tfrac{x}{e^x}}_{\to 0}+1-\underbrace{\tfrac{e}{e^x}}_{\to 0}}_{\to 1}\big)\Big]$$
$$=\infty$$

$$\lim_{x\to-\infty} f(x) = \infty + 0 - e = \infty$$

$\begin{cases} f(0)<0 \\ \lim_{x\to\infty} f(x)>0 \end{cases} \Rightarrow$ at least one solution $f(x)=0$ in $[0,\infty)$

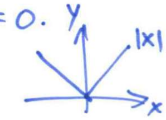
$\begin{cases} \lim_{x\to-\infty} f(x)>0 \\ f(0)<0 \end{cases} \Rightarrow$ at least one solution $f(x)=0$ in $(-\infty, 0]$

Max / Min for single variable. functions

~~Discon~~ Candidates: discontinuity; endpoints (boundry points); critical points.

Note: • discontinuity and isolated points are not too interesting in Math 2.

• nondifferentiability points that are not boundry are not too important, either. though $|x|$ has minimum at $x=0$.



• endpoints — must check!

• stationary points — important!

Ex: $0\le x\le 1$. $\max\ (\frac{x}{4})^2+(\frac{1-x}{4})^2 = \frac{1}{16}(2x^2-2x+1)=f(x)$

• ~~End~~ Check the endpoints.

$f(0)=\frac{1}{16}=f(1) \Rightarrow \max$

• Check the critical points.

$f'(x)=\frac{1}{16}(4x-2)=0 \Rightarrow x=\frac{1}{2}$

$f(\frac{1}{2})=\frac{1}{32} \Rightarrow \min$