Lecture Note 5-6

Single-variable optimization

Ex: ~~$f(x)=x^2$~~ $f(x)=-x^2$

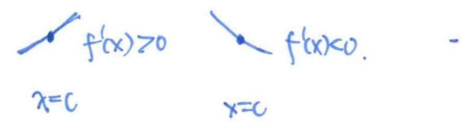
guess and verify that $x=0$ is the max.

Necessary first-order condition.

f is differentiable in an interval I, c is an interior point.

for $x=c$ to be max/min in I $\Rightarrow$ $f'(x)=0$.

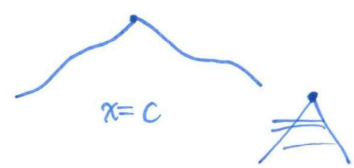
i.e., $x=c$ is a critical point.



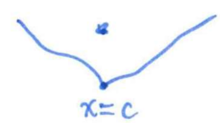
First-derivative test.

f is differentiable in I, which includes c.

(i) $\left.\begin{array}{ll} f'(x) \geq 0 & \text{for } x \leq c \\ f'(x) \leq 0 & \text{for } x \geq c \end{array}\right\} \Rightarrow x=c$ is a max in I



(ii) $\left.\begin{array}{ll} f'(x) \leq 0 & \text{for } x \leq c \\ f'(x) \geq 0 & \text{for } x \geq c \end{array}\right\} \Rightarrow x=c$ is a min in I

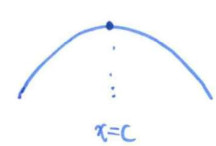


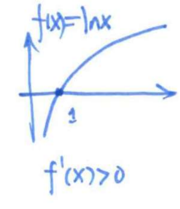
concave and convex

c is a critical point, and interior of I.

(i). If f is concave, c is max in I.

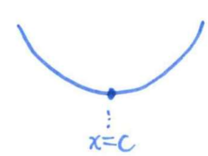
$f''(x) \leq 0$





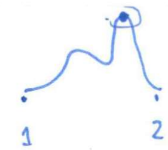
(ii). If f is convex, c is min in I

$f''(x) \geq 0$



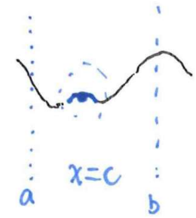
Sometimes need to check the end points.

Local max/min



First-Direvative test.

Second-Drrevative test



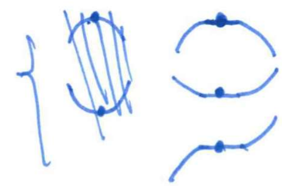
$x=c$ is a strict local max but not a max in $[a,b]$.

(i). $\left.\begin{array}{l} f'(c)=0 \\ f''(c)<0 \end{array}\right\} \Rightarrow x=c$ strict local max

(ii). $\left.\begin{array}{l} f'(c)=0 \\ f''(c)>0 \end{array}\right\} \Rightarrow x=c$ strict local min

(iii) $\left.\begin{array}{l} f'(c)=0 \\ f''(c)=0 \end{array}\right\} \Rightarrow$ undetermined.

why? $f''(c)=0 \Rightarrow$



Multivariable optimization.

two-variable

- ~~Nec~~ Necessary first-order conditions:

  want to find max/min of $f(x,y)$

  an interior point (x_0,y_0) could be the solution only if

  $$f_1'(x,y)=0, \quad f_2'(x,y)=0.$$

- sufficient conditions for max/min

  If $f_{11}''(x,y)\le 0$; $f_{22}''(x,y)\le 0$
  $f_{11}''(x,y)f_{22}''(x,y)-[f_{12}''(x,y)]^2 \ge 0$. for all (x,y)

  then the solution of the FOCs is the (global) max.

  If $f_{11}''(x,y)\ge 0$; $f_{22}''(x,y)\ge 0$
  $f_{11}''(x,y)f_{22}''(x,y)-[f_{12}''(x,y)]^2 \ge 0$ for all (x,y)

  then the solution of the FOCs is the (global) min.

Local max/min

second - derivative test

if $f_1'(x_0, y_0) = 0 \quad f_2'(x_0, y_0) = 0$

$f_{11}''(x_0, y_0) < 0$; $f_{22}''(x_0, y_0) < 0$ for (x_0, y_0)

$f_{11}''(x_0, y_0) f_{22}''(x_0, y_0) - [f_{12}''(x_0, y_0)]^2 > 0$

then (x_0, y_0) is the strict local max.

if $f_{11}'' > 0$; $f_{22}'' > 0$ for (x_0, y_0).

$f_{11}'' f_{22}'' - (f_{12}'')^2 > 0$

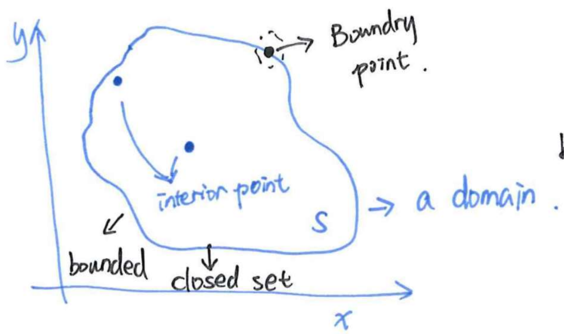
then → strict local min

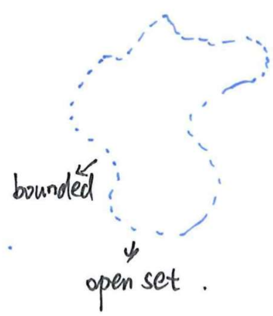
if $f_{11}'' f_{22}'' - (f_{12}'')^2 < 0$.

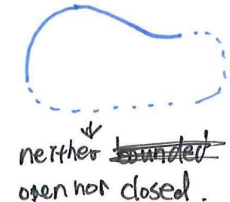
then (x_0, y_0) is a saddle point.

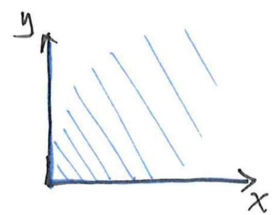
if $f_{11}'' f_{22}'' - (f_{12}'')^2 = 0$, then (x_0, y_0) could be max, min or saddle point.

The Extreme value theorem









the domain
$x \geq 0$, $y \geq 0$
is unbounded, even though it is closed.

Suppose $f(x,y)$ is continuous throughout a nonempty, closed, and bounded set S, then there exists a max, and a min.

Proof by contradiction (a small digression).

Assume what you want to disprove

Then deduce a contradiction.

Ex: Let n be an natural number, so that n^2 is even, prove that n is even.

Proof: Assume n is odd, so ~~n=2k~~ $n = 2k-1$

then $n^2 = (2k-1)^2 = 4k^2 - 4k + 1$

$= \underbrace{\underbrace{4(k^2-k)}_{\text{even}} + 1}_{\text{odd}}$ → contradiction.

so n can not be odd

could only be even

Constrained optimization

$$\max(\min)\ f(x,y) \quad \text{s.t.} \quad g(x,y)=c$$

The Lagrange method.

Step 1: write down the Lagrangian.

$$\mathcal{L}(x,y) = f(x,y) - \lambda[g(x,y)-c]$$

Step 2: Differentiate $\mathcal{L}(x,y)$ wrt x, y.

$$\begin{cases} \dfrac{\partial \mathcal{L}(x,y)}{\partial x} = f_1'(x,y) - \lambda g_1'(x,y) = 0 \\ \dfrac{\partial \mathcal{L}(x,y)}{\partial y} = f_2'(x,y) - \lambda g_2'(x,y) = 0 \\ g(x,y) = c \end{cases}$$

Step 3: we have three equations in three unknowns.

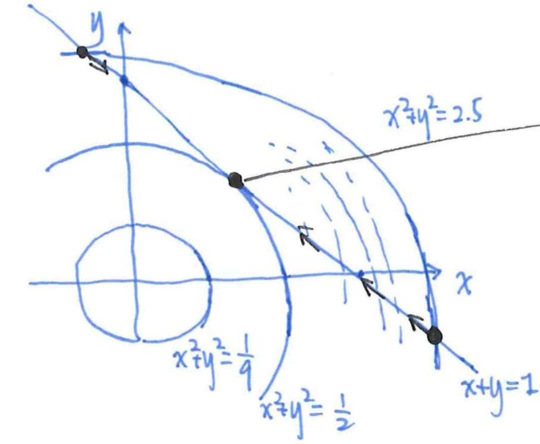
solve.

Note: if $g_1'(x,y)$ and $g_2'(x,y)$ both vanish ~~vanishes~~, then the Lagrange method might fail

Why does it work?

Geometric. $f(x,y)$ $g(x,y)=1$

Consider $\min\ x^2+y^2$ s.t. $x+y=1$



implicit differentiation

$g(x,y)=1$

$\Longrightarrow g_1'(x,y)+g_2'(x,y)\frac{dy}{dx}=0$.

slope of the line: $\frac{dy}{dx}=-\frac{g_1'(x,y)}{g_2'(x,y)}$

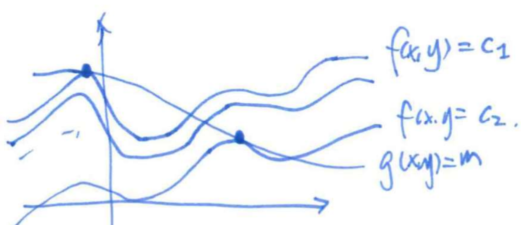
$||$

slope of the level curve $x^2+y^2=\frac{1}{2}$

$$\frac{dy}{dx}=-\frac{f_1'(x,y)}{f_2'(x,y)}$$

$$\Rightarrow -\frac{f_1'(x,y)}{f_2'(x,y)}=-\frac{g_1'(x,y)}{g_2'(x,y)}$$

$$\Rightarrow \frac{f_1'}{g_1'}=\frac{f_2'}{g_2'}=\lambda \Rightarrow \begin{array}{l} \cancel{f_1'}\ f_1'-\lambda g_1'=0 \\ f_2'-\lambda g_2'=0 \end{array}$$



Intepreting the Lagrange multiplier "λ"

$$\max(\min)\ f(x,y)\ \text{ s.t. } g(x,y)=c$$

Suppose x^* and y^* are the solution with λ^*.

Then the optimal $f(x,y)$ would be $f(x^*,y^*)=f^*$.

If c changes slightly $\Rightarrow$ x^*, y^* will also change

$\Rightarrow x^*=x^*(c)$, $y^*=y^*(c)$, $f^*(c)=f(x^*(c),y^*(c))$

$$\frac{df^*(c)}{dc}=\lambda^*.$$

we can also have:

$$f^*(c+dc)-f^*(c)\approx\lambda^* dc.$$

Deduction: when increase c by "dc"

$$\frac{df^*(c)}{dc}=\lambda^* \Rightarrow df^*(c)=\lambda\, dc$$

$$df^*(c) \approx f^*(c+dc) - f^*(c)$$

$$\Rightarrow f^*(c+dc) - f^*(c) \approx \lambda dc.$$

- <u>More than 2 variables</u>

Ex: max $f(x,y,z) = 4z - x^2 - y^2 - z^2$, s.t. $g(x,y,z) = z - xy = 0$

Step 1: $\mathcal{L}(x,y,z) = 4z - x^2 - y^2 - z^2 - \lambda(z - xy)$

Step 2:
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = -2x + \lambda y = 0 & \cdots\cdots (1) \\ \frac{\partial \mathcal{L}}{\partial y} = -2y + \lambda x = 0 & \cdots\cdots (2) \\ \frac{\partial \mathcal{L}}{\partial z} = 4 - 2z - \lambda = 0 & \cdots\cdots (3) \\ z - xy = 0 & \cdots\cdots (4) \end{cases}$$

$(3) + 2\times(4) \Rightarrow \cancel{4 - \lambda - 2y}\ 4 - \lambda - 2xy = 0 \Rightarrow \lambda = 4 - 2xy. \cdots (5)$

substituting (5) into (1) and (2)

$2x = (4-2xy)y \Rightarrow x = \frac{1}{2}(4-2xy)y \quad \cdots (6)$

$2y = (4-2xy)x \quad \cdots (7)$

substituting (6) into (7).

$$2y = \frac{1}{2}(4-2xy)^2 y$$

$$\Rightarrow y = 0 \quad \text{or} \quad \underbrace{4 = (4-2xy)^2}$$

$$\Rightarrow 4 - 2xy = \pm 2$$

(i). $y = 0 \Rightarrow z - xy = 0 \Rightarrow z = 0 \xrightarrow{4-2z-\lambda=0} \lambda = 4$

$\Rightarrow 2x = \lambda y \Rightarrow x = 0$

solution candidate $(x,y,z) = (0,0,0)$, with $\lambda = 4$. $f(0,0,0) = 0$.

(ii) $4 - 2xy = 2 \Rightarrow xy = 1 \xrightarrow{(4)} z = 1 \xrightarrow{(3)} \lambda = 2 \xrightarrow{(1)} x = y$

$\Rightarrow x = y = 1$ or $x = y = -1$

solution candidates: $(x,y,z) = (1,1,1)$ with $\lambda = 2$, $f(1,1,1) = 1$

$(x,y,z) = (-1,-1,1)$ with $\lambda = 2$, $f(-1,-1,1) = 1$

(iii) $4-2xy=-2 \Rightarrow xy=3 \xrightarrow{④} z=3 \xrightarrow{③} \lambda=-2$

$\xrightarrow{①} x=-y \longrightarrow -x^2=3 \Rightarrow x^2=-3 \Rightarrow$ invalid.

By comparison, we see that the solution candidate $(1,1,1)$ and $(-1,-1,1)$ solve the problem. (with $f^*=1$, $\lambda^*=2$)

Note: if we change the constraint to $z-xy=0.02$

$$f^*(\text{new}) - f^*(\text{old}) = 0.02\,\lambda^* = 0.02\times 2 = 0.04.$$

- **More than 1 constraint**

Ex: $\min\ x^2+y^2+z^2$, s.t. $\begin{cases} x+2y+z=30 \\ 2x-y-3z=10 \end{cases}$

step 1: $\mathcal{L} = x^2+y^2+z^2 - \lambda_1(x+2y+z-30) - \lambda_2(2x-y-3z-10)$

step 2: $\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z} = 0 \\ x+2y+z=30 \\ 2x-y-3z=10 \end{cases}$

step 3: We have 5 equations in 5 unknowns. solve.