Lecture 7-10

Nonlinear Programming. (inequality constraints)

$$\max f(x,y) \quad s.t\ g(x,y) \le c$$

The Kuhn–Tucker method.

i) $\mathcal{L}(x,y) = f(x,y) - \lambda[\underbrace{g(x,y)-c}_{\le 0}]$

$\rightarrow$ If I have $g(x,y) \ge c$ Then $\mathcal{L} = \dots - \lambda(c - g(x,y))$

The Kuhn–Tucker conditions.

ii) $\mathcal{L}'_1(x,y) = f'_1(x,y) - \lambda g'_1(x,y) = 0$

$\mathcal{L}'_2(x,y) = f'_2(x,y) - \lambda g'_2(x,y) = 0$

iii) $\lambda \ge 0$, with $\lambda = 0$ if $g(x,y) < c$

or $\lambda \ge 0$ and $\lambda(g(x,y) - c) = 0 \Rightarrow$ complementary slackness condition.

iv) $g(x,y) \le c$

The points (x,y), ~~#~~ with associated λ, that solve ii), iii) and iv), are our solution candidates.

Ex: $\max x^2 + y^2 \quad s.t.\ (x-3)^2 + (y-2)^2 \le 9$

i) $\mathcal{L} = x^2 + y^2 - \lambda[(x-3)^2 + (y-2)^2 - 9]$

ii) $\mathcal{L}'_1 = 2x - 2\lambda(x-3) = 0$ ①

$\mathcal{L}'_2 = 2y - 2\lambda(y-2) = 0$ ②

iii) $\lambda \ge 0$, $\lambda[(x-3)^2 + (y-2)^2 - 9] = 0$. ③

iv) $(x-3)^2 + (y-2)^2 \le 9$ ④.

~~case~~ start with ③.

case $\lambda = 0 \Rightarrow$ in ① $2x = 0 \Rightarrow x = 0$

in ② $y = 0$.

sub. $x = y = 0$ into ④ $\Rightarrow 9 + 4 > 9 \Rightarrow$ not a solution.

case $\lambda > 0 \Rightarrow (x-3)^2 + (y-2)^2 = 9$ ⑤

Now look at ① $2x = 2\lambda(x-3)$

② $2y = 2\lambda(y-2)$

want to divide ① by ② to cancel λ. need $y \neq 0$, $y-2 \neq 0$.

check : if $y=0 \overset{②}{\Rightarrow} 0 = \underbrace{-4\underset{>0}{\lambda}}_{\neq 0} \Rightarrow$ contradiction.

if $y-2=0 \Rightarrow 4 = 2\lambda \cdot 0 = 0. \Rightarrow$ contradiction.

Divide ① by ② : $\frac{x}{y} = \frac{x-3}{y-2}$

$\Rightarrow x(y-2) = (x-3)y.$

$\Rightarrow \cancel{xy} - 2x = \cancel{xy} - 3y$

$\Rightarrow 3y = 2x \Rightarrow y = \frac{2}{3}x \quad \ldots ⑥$

sub. ⑥ into ⑤ : $(x-3)^2 - (\frac{2}{3}x - 2)^2 = 9.$

Sufficient conditions (for the Kuhn-Tucker)

Suppose you have a constrained optimization problem with inequality constraints,
maximization

Then suppose (x_0, y_0) is the solution candidate given by the Kuhn-Tucker method.

~~write~~ with the Lagrangian:

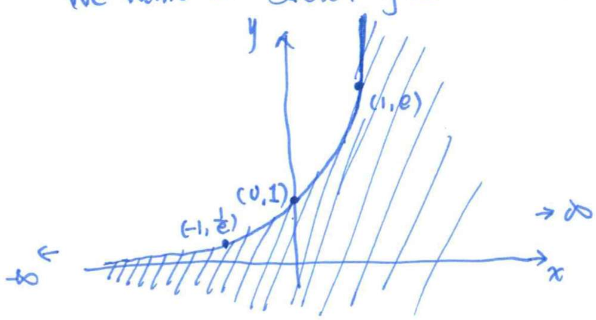
$$\mathcal{L}(x,y) = f(x,y) - \lambda[g(x,y) - c]$$

If the Lagrangian is concave, then (x_0, y_0) is the max.

Ex: $\max\ 1-x^2$ s.t. $y - e^x \leq 0$

(a) sketch the domain.

we want to sketch $y \leq e^x \Rightarrow$ sketch $y = e^x$



(b) write down the K-T conditions.

i) $\mathcal{L}(x,y) = 1 - x^2 - \lambda(y - e^x)$

ii) $\mathcal{L}'_x = -2x + \lambda e^x = 0$
$\mathcal{L}'_y = -\lambda = 0$

iii) $\lambda \geq 0$, $\lambda(y - e^x) = 0$

iv) $y \leq e^x$

K-T conditions.

(c) find the points that satisfy K-T.

standard: ~~Ex~~ Discuss: $\lambda > 0$
$\lambda = 0$

This is not necessary here since we have $\lambda = 0$ from ii) which give us $x=0$, $y \leq e^x$.

or Discuss $y \leq e^x$:
$y < e^x$
$y = e^x$

From (ii): $\lambda = 0 \Rightarrow x = 0 \Rightarrow y \leq e^0 = 1$.

Then $(0,t)$ are solution candidates with $\lambda = 0$, $t \leq 1$

(d). Find the solution if it exists.

Method 1: evaluating the objective function at all the candidates, then compare them. But you need to prove that max exists.

Try extreme value theorem: our domain is not bounded, which means that the extreme value theorem does not ~~gu~~ guarantee a max.

$\Rightarrow$ can not use this method here.

Method 2:

The sufficient conditions.

$\mathcal{L} = 1 - x^2 - \lambda(y - e^x)$

Plug in $\lambda = 0 \Rightarrow \mathcal{L} = 1 - x^2$ which is concave in (x,y). $\cancel{\mathcal{L}''_{xx} = -2 < 0}$ $\cancel{\mathcal{L}''_{yy} = 0 \leq 0}$

so our solution candidates solve the problem.

Multiple Inequality Constraints

several variables / constraints.

- One multiplier for each constraints.
- Complementary slackness for each multiplier / constraint.

$$\max f(x_1, x_2, \ldots x_n) \text{ s.t } \begin{cases} g_1(x_1, \ldots, x_n) \leq c_1 \\ \vdots \\ g_m(x_1, \ldots, x_n) \leq c_m. \end{cases}$$

(i) $\mathcal{L} = f(x_1, \ldots, x_n) - \sum_{j=1}^{m} \lambda_j (g_j - c_j)$

(ii) $\mathcal{L}'_{x_i} = 0$ for $i = 1, \ldots, n$

(iii) $\lambda_j \geq 0$, with $\lambda_j = 0$ if $g_j(\vec{x}) < c_j$ for all j

or $\lambda_j (g_j(\vec{x}) - c_j) = 0$

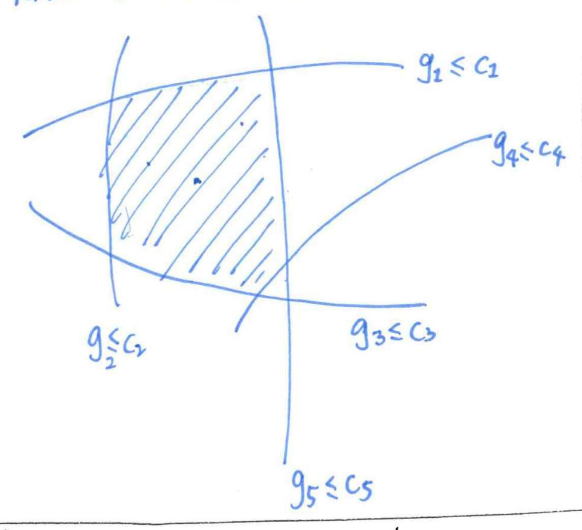
(iv) $g_j(\vec{x}) \leq c_j$ for all j.

Sufficient conditions: $\mathcal{L}$ concave after plugging in all the λ_j.

Note: If I want to make a Lagrange problem with 2 constraints, I would need at least 3 variables.

But for K-T problem, I could have any amount of constraints in two variable.

Multi-constraints sketch



The possible points of solution, are:

- Interior point. $\Rightarrow$ no contraints bind
- on one of the five curve segments. $\Rightarrow$ one constraint binds. 4 not bind.
- on one of the five intersection points between two of the five curves. $\Rightarrow$ two constraints bind, 3 not bind.

$\Rightarrow$ only need to discuss 11 cases, $\Rightarrow$ better than 32 cases

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Terminology: "active" / "binding" constaint: one that holds with "="

often a good idea to discuss λ.

$\lambda > 0$ or $\lambda = 0$ for each λ.

binding contraint. possibly binding

or we discuss the case where the constraint is "binding" or "not binding"

$g(\vec{x}) = c$ $\quad$ $g(\vec{x}) < c$

Looking at the sketch: we have five constraints $\Rightarrow$ 5 complementary slackness

For each constraint, we have 2 cases.

Then for 5 constraints $\Rightarrow$ $2^5 = 32$ cases. $\Rightarrow$ too many.

For multiple constraints, it's often easier to look at the graph.

Ex: $\max \frac{1}{2}(x^2+y^2)$ s.t. $y \le 1 - a(x-1)$

$x \geq 0$ $\quad$ with $a > 0$.

$y \geq 0$

Q: Find all points that satisfy K-T.

(i). $\mathcal{L} = \frac{1}{2}(x^2+y^2) - \lambda_1 [y + a(x-1) - 1] - \lambda_2(-x) - \lambda_3(-y)$

(ii) $\mathcal{L}'_x = x - a\lambda_1 + \lambda_2 = 0$ $\quad \cdots$ ①

$\mathcal{L}'_y = y - \lambda_1 + \lambda_3 = 0$ $\quad \cdots$ ②

(iii) $\lambda_1 \geq 0, \quad \lambda_1(y + a(x-1) - 1) = 0 \quad \cdots ③$

$\lambda_2 \geq 0, \quad \lambda_2(-x) = 0 \quad \cdots ④$

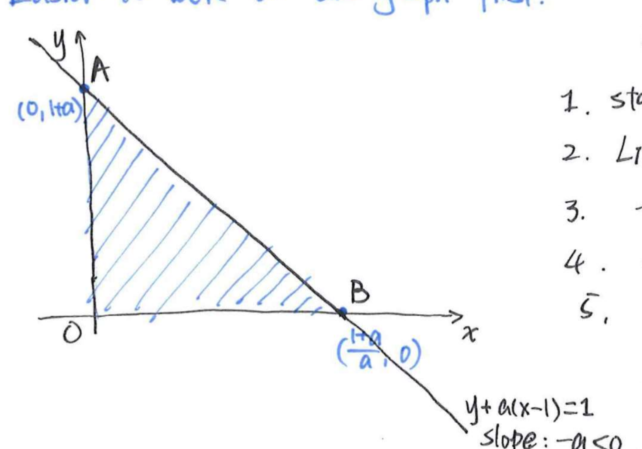
$\lambda_3 \geq 0, \quad \lambda_3(-y) = 0 \quad \cdots ⑤$

(IV) $y + a(x-1) \leq 1 \quad \cdots ⑥$

$-x \leq 0 \quad \cdots ⑦$

$-y \leq 0 \quad \cdots ⑧$

Easier to look at the graph first.



Possible cases that satisfy K-T.

1. stationary point within the area AOB.
2. Line segment OB, not including point. O, B
3. 〃 OA, 〃 O, A
4. 〃 AB, 〃 A, B
5. The corner A
6. 〃 B
7. 〃 O

case 1, 7 $\lambda_1 = \lambda_2 = \lambda_3 = 0 \xRightarrow{①,②} \underline{\underline{(x, y) = (0, 0)}}$

~~case 2, 6 Line OB, excl. O~~

$y = 0, x > 0 \xRightarrow{④} \lambda_2 = 0 \Rightarrow (x, y) = (0, 0)$

$\xRightarrow{②} \lambda_1 = \lambda_3$ either $\lambda_1 = \lambda_3 = 0$

or $\lambda_1 = \lambda_3 > 0$

$\xRightarrow{③} y = 1 - a(x-1) \xRightarrow{y=0} x = \frac{1+a}{a} \Rightarrow \left(\frac{1+a}{a}, 0\right)$

It's ok to get (0, 0) here even though we "excluded" point O.

case 2.6. Line OB, excl. O

$\underline{\underline{y = 0}}, x > 0 \xRightarrow{④} \lambda_2 = 0 \xRightarrow{①} x = a\lambda_1 > 0 \Rightarrow \lambda_1 > 0$

$\xRightarrow{②} \lambda_1 = \lambda_3 \Rightarrow \lambda_1 = \lambda_3 > 0 \xRightarrow{③} y = 1 - a(x-1) \xRightarrow{y=0} x = \frac{1+a}{a} \Rightarrow \underline{\underline{\left(\frac{1+a}{a}, 0\right)}}$

case 3, 5 Line OA, excl. O

$x = 0, y > 0 \xRightarrow{⑤} \lambda_3 = 0 \xRightarrow{②} y = \lambda_1 > 0$

$\xRightarrow{①} a\lambda_1 = \lambda_2 \Rightarrow a\lambda_1 = \lambda_2 > 0 \xRightarrow{③} y + a(x-1) = 1 \xRightarrow{x=0} y = 1+a$

$\Rightarrow \underline{\underline{(0, 1+a)}}$

case 4. Line AB, excl. A.B

$x>0,\ y>0 \underset{④⑤}{\Longrightarrow} \lambda_2=\lambda_3=0$

From ① $x=a\lambda_1$
② $y=\lambda_1$ $\Big\}$ $x=ay$

We are on line AB $\Rightarrow$ $y+a(x-1)-1=0 \xrightarrow{y=\frac{x}{a}} \frac{x}{a}+a(x-1)-1=0$

$\Rightarrow x=\frac{a+a^2}{1+a^2} \Rightarrow y=\frac{1+a}{1+a^2}$

$\Rightarrow (x,y)=\left(\frac{a+a^2}{1+a^2},\ \frac{1+a}{1+a^2}\right)$

Connection between λ and binding constraint.

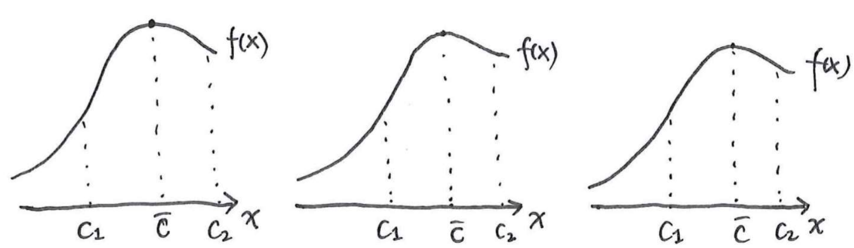
When we assume $\lambda=0$, we are testing the case where the corresponding constraint is "$g\le c$"

... assume $\lambda>0$, ... "$g=c$"

In other words. $\lambda>0 \Rightarrow g=c$

$\lambda=0 \Rightarrow g=c$ or $g<c$

Why?



max $f(x)$. s.t. $x\le c$

$c=c_1$	$c=c_2$	$c=\bar{c}$
$x^*=c_1$.	$x^*=\bar{c}$	$x^*=\bar{c}$
$f(x^*)=f^*(c_1)$	$\lambda=0$	$\lambda=0$
$\lambda>0$	$g<c$	$g=c$

$\lambda>0 \Rightarrow g=c$

$\lambda=0$ could imply $g<c$, or $g=c$

<u>Envelope theorem</u>

$$\max_{x} f(\overset{\text{variable}}{x}, \underset{}{r}) \xrightarrow{} \text{parameter}$$

for different r, we get different x^*

Let x^* be the solution, easy to see that $x^* = x^*(r)$

Then we have the <u>value function</u>:

$$f^*(r) = f(x^*(r), r)$$

$$\frac{df^*(r)}{dr} = \underbrace{f_1'(\cancel{x^*}\, x^*(r), r)}_{=0} \frac{dx^*(r)}{dr} + f_2'(x^*(r), r)$$

because x^* max $f(x,r) \Rightarrow f_1'(x^*, r) = 0$.

$$\Rightarrow \frac{df^*(r)}{dr} = f_2'(x^*(r), r)$$

- It only works on the value function / optimized objective function

Ex: $\max_{x} \pi(x, r) = rx - x^2$

FOC: $r - 2x = 0 \Rightarrow x = \frac{r}{2}$

2-derivative: $-2 < 0 \Rightarrow$ ~~strict~~ π is strictly concave $\Rightarrow x^* = \frac{r}{2}$ is global max

Then $\pi(x^*(r), r) = r \cdot \frac{r}{2} - \frac{r^2}{4} = \frac{1}{4} r^2 = \pi^*(r)$

$$\frac{d\pi^*(r)}{dr} = \frac{r}{2}.$$

Envelop theorem: $\pi(x^*(r), r) = rx^* - (x^*)^2 = \pi^*(r)$

$$\frac{d\pi^*(r)}{dr} = \cancel{x^*}\ \pi_2'(x^*(r), r) = x^*.$$

<u>Multivariable — Envelope theorem</u>

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ n variables

$\vec{r} = (r_1, r_2, \ldots, r_m)$ m parameters.

want to $\max_{\vec{x}} f(\vec{x}, \vec{r})$.

Let $\vec{x}^*(\vec{r})$ be the value of $\vec{x}$ that maximizes $f(\vec{x}, \vec{r})$

easy to see that $\vec{x}^* = \vec{x}^*(\vec{r})$, $f(\vec{x}^*(\vec{r}), \vec{r}) = f^*(\vec{r})$

Then $$\frac{\partial f^*(\vec{r})}{\partial r_j} = \frac{\partial f(\vec{x}^*(\vec{r}), \vec{r})}{\partial r_j}$$

provided that the partial derivatives exist.

constrained - envelope.

$$\max(\min)\ f(x,y) \quad s.t.\ g(x,y)=c$$

Let x^*, y^* be the solution candidates with corresponding $\lambda(c)$

$$f^*(c) = f(x^*(c), y^*(c))$$

$$\frac{df^*(c)}{dc} = \lambda(c)$$

Proof[1]: $\frac{df^*(c)}{dc} = \frac{\partial f(x^*,y^*)}{\partial x^*}\frac{\partial x^*(c)}{\partial c} + \frac{\partial f(x^*,y^*)}{\partial y^*}\frac{\partial y^*(c)}{\partial c}$

recall that we get x^*, y^* by solving the Lagrange FOCs.

so: $\frac{\partial f(x^*,y^*)}{\partial x^*} - \lambda\frac{\partial g(x^*,y^*)}{\partial x^*} = 0$

$\frac{\partial f(x^*,y^*)}{\partial y^*} - \lambda\frac{\partial g(x^*,y^*)}{\partial y^*} = 0.$

$\hookrightarrow \frac{df^*(c)}{dc} = \lambda\left[g_1'(x^*,y^*)\frac{\partial x^*(c)}{\partial c} + g_2'(x^*,y^*)\frac{\partial y^*(c)}{\partial c}\right]$

since $g(x^*,y^*)=c \xrightarrow[\text{sides wrt } c]{\text{Differentiating both}} g_1'(x^*,y^*)\frac{\partial x^*(c)}{\partial c} + g_2'(x^*,y^*)\frac{\partial y^*(c)}{\partial c} = 1.$

$\hookrightarrow \frac{df^*(c)}{dc} = \lambda$

Proof[2]: $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$

$$\left\{\begin{array}{l} \frac{\partial\mathcal{L}}{\partial x} = f_1' - \lambda g_1' = 0 \\ \frac{\partial\mathcal{L}}{\partial y} = f_2' - \lambda g_2' = 0 \quad \Rightarrow x^*, y^*, \lambda^*. \\ \qquad\qquad\qquad\qquad\qquad \Downarrow \\ \qquad\qquad\qquad\qquad\qquad x^*(c),\ y^*(c),\ \lambda^*(c) \\ \frac{\partial\mathcal{L}}{\partial\lambda} = g(x,y) - c = 0 \end{array}\right.$$

$$f^*(c) = f(x^*,y^*) = f(x^*,y^*) - \lambda^*\underbrace{(g(x^*,y^*)-c)}_{=0.} = \mathcal{L}(x^*,y^*,\lambda^*) = \mathcal{L}^*(c)$$

$$\frac{df^*(c)}{dc} = \frac{d\mathcal{L}^*(c)}{dc} = \frac{\partial\mathcal{L}(x^*,y^*,\lambda^*)}{\partial c} = \lambda^*.$$

General case for Lagrange multiplier

Let $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{c} = (c_1, c_2, \ldots, c_m)$

$\max(\min)\ f(\vec{x})$ s.t. $g_j(\vec{x}) = c_j$, $j = 1, 2, \ldots, m$.

Let $\vec{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ satisfy the necessary conditions.

Assume each $x_i^*(\vec{c})$ is differentiable.

Then we have $f^*(\vec{c}) = f(\vec{x}^*(\vec{c}))$

$$\frac{\partial f^*(\vec{c})}{\partial c_j} = \lambda_j(\vec{c})$$

sketch proof: $f^*(\vec{c}) = \mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = \mathcal{L}^*(\vec{c})$

$= f^*(\vec{x}^*) - \lambda_1^*(g_1(\vec{x}^*) - c_1) - \lambda_2^*(g_2(\vec{x}^*) - c_2) - \ldots$

$\ldots - \lambda_j^*(g_j(\vec{x}^*) - c_j) - \ldots - \lambda_m^*(g_m(\vec{x}^*) - c_m)$

$$\frac{df^*(\vec{c})}{dc_j} = \frac{d\mathcal{L}^*(\vec{c})}{dc_j} = \frac{\partial \mathcal{L}(\vec{x}^*, \vec{\lambda}^*)}{\partial c_j} = \lambda_j^*(\vec{c})$$

Note: Without the Envelope theorem, you will have to differentiate ~~all the~~ $f(\vec{x}^*)$ wrt all the x^*s, and all the $g_j(\vec{x}^*)$ wrt all the x^*s, and all the $\lambda^*(\vec{c})$ wrt c_j, ... to find $\frac{df^*(\vec{c})}{dc_j}$

But with the envelope theorem, it's only one step.

The (General) Envelope theorem

$\max(\min)_{\vec{x}}\ f(\vec{x}, \vec{r})$ s.t. $g_j(\vec{x}, \vec{r}) = 0$, for $j = 1, 2, \ldots, m$

$$\mathcal{L}(\vec{x}, \vec{r}) = f(\vec{x}, \vec{r}) - \sum_{j=1}^{m} \lambda_j g_j(\vec{x}, \vec{r})$$

Let $\vec{x}^*$ be the solution (candidates).

$$f^*(\vec{r}) = f(\vec{x}^*(\vec{r}), \vec{r})$$

If $f^*(\vec{r})$ and $\vec{x}^*(\vec{r})$ are differentiable.

$$\frac{\partial f^*(\vec{r})}{\partial r_h} = \frac{\partial \mathcal{L}(\vec{x}^*(\vec{r}), \vec{r})}{\partial r_h}$$

for each $h = 1, 2, \ldots, k$.

Ex: 2007 Autumn #4.

$$\max_{(x,y)} \; 4e^x + \frac{1}{2}Ax^2y^2 + e^{3y} \quad \text{s.t.} \quad \begin{cases} x^2 + By^2 \leqslant C \\ x \geqslant 0 \\ y \geqslant 0. \end{cases}$$

A, B, C are strictly positive constraints.

(a). State the K-T.

$$\mathcal{L} = 4e^x + \frac{1}{2}Ax^2y^2 + e^{3y} - \lambda_1(x^2 + By^2 - C) - \lambda_2(-x) - \lambda_3(-y)$$

$$\mathcal{L}'_x = 4e^x + Axy^2 - 2\lambda_1 x + \lambda_2 = 0 \quad \cdots ①$$

$$\mathcal{L}'_y = Ax^2y + 3e^{3y} - 2B\lambda_1 y + \lambda_3 = 0 \quad \cdots ②$$

$$\lambda_1 \geqslant 0, \quad \lambda_1(x^2 + By^2 - C) = 0 \quad \cdots ③$$

$$\lambda_2 \geqslant 0, \quad \lambda_2(-x) = 0 \quad \cdots ④$$

$$\lambda_3 \geqslant 0, \quad \lambda_3(-y) = 0 \quad \cdots ⑤$$

conditions ①–⑤ are our K-T.

(b) show that the K-T imply $x^2 + By^2 = C$ and $xy \neq 0$.

Note: Do not try to solve the problem.

Let's assume $x^2 + By^2 < C \xrightarrow{③} \lambda_1 = 0 \xrightarrow{①} \underbrace{\underbrace{4e^x}_{>0} + \underbrace{Axy^2}_{\geqslant 0} + \underbrace{\lambda_2}_{\geqslant 0}}_{>0} = 0 \Rightarrow$ contradiction $\Downarrow$ $x^2 + By^2 = C$

Let's assume $x = 0 \xrightarrow{①} 4 + \lambda_2 = 0 \Rightarrow \lambda_2 = -4 \Rightarrow$ contradiction

Let's assume $y = 0 \xrightarrow{②} 3 + \lambda_3 = 0 \Rightarrow \lambda_3 = -3 \Rightarrow$ contradiction

$\Big\}$ $x \neq 0,\ y \neq 0.$

$xy \neq 0$

Sth more on Envelope theorem

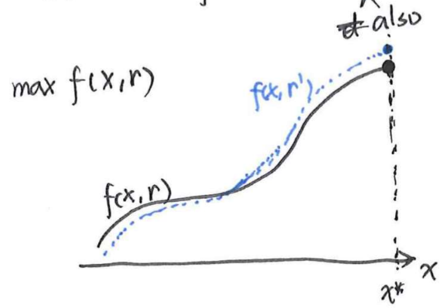
For details, see notes from before.

Now look at the sketch proof of Envelope.

$$\frac{d f^*(r)}{dr} = \underbrace{\frac{\partial f(x^*(r), r)}{\partial x}}_{=0 \text{ at } x=x^*} \frac{\partial x^*(r)}{\partial r} + \frac{\partial f(x^*(r), r)}{\partial r}$$

works for all stationary points (max, min, saddle points)

The Envelope theorem also works if x^* is an endpoint. (at the boundries).



when r changes to r'. if the change is not too large, then the shape of the function curve would not change too much, and we achieve max at the same endpoint.

$$\frac{d f^*(r)}{dr} = \underbrace{\frac{\partial f(x^*, r)}{\partial x}}_{>0 \text{ from the left}} \underbrace{\frac{\partial x^*(r)}{\partial r}}_{=0 \text{ at the endpoint.}} + \frac{\partial f(x^*, r)}{\partial r}$$

Bottom line: When you want to use the Envelope theorem, you don't need to check whether the optimal points are interior or endpoints.

Envelope theorem holds for:

- max or min, whether interior stationary or endpoints.
- saddle points.

May fail if optimum is not unique.

Envelope in K-T

$$\max\ f(x,y) \quad s.t.\ g(x,y) \le c$$

$$\mathcal{L} = f(x,y) - \lambda(g(x,y) - c)$$

$$\Rightarrow (x^*, y^*) = (x^*(c), y^*(c))$$

$$\Rightarrow f^*(c) = f(x^*(c), y^*(c))$$

$$\frac{d f^*(c)}{dc} = \frac{\partial \mathcal{L}(x^*(c), y^*(c), \lambda^*(c))}{\partial c} = \lambda^*(c)$$

why?

we either have $g(x^*, y^*) = c$, same as in the Lagrange problem.

or have $g(x^*, y^*) < c$, • in this case when c increases, nothing happens to x^*, y^*, and also $f(x^*, y^*)$.

$\Rightarrow \frac{d f^*(c)}{dc} = 0$.

• $\lambda = 0$ by complementary slackness.

It is possible to have both $\lambda = 0$ and $g = c$ if we let $g = c$ passes through a local maximum.

Ex: $\max (-x^2)$ s.t. $x \leq 0$.

$\mathcal{L} = -x^2 - \lambda x$

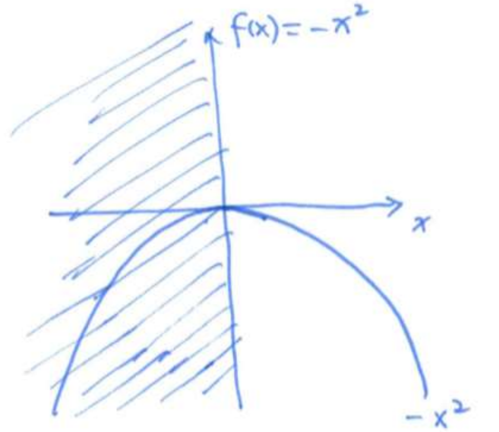
$\mathcal{L}'_x = -2x - \lambda = 0 \quad \cdots ①$

$\lambda \geq 0, \ \lambda x = 0 \quad \cdots ②$

From ① $\Rightarrow \lambda = -2x \xrightarrow{②} \cancel{(2x)}$ $(-2x) \cdot x = 0$

$\Rightarrow -2x^2 = 0 \Rightarrow x^* = 0$.

$\lambda^* = -2x^* = 0$.



Note: Consider $\max f(x,y)$ s.t. $g(x,y) \leq c$.

if $f(x,y)$ have stationary point(s), for example, (x_0, y_0)

such that (x_0, y_0) lies within the set $g(x,y) \leq c$, or (is interior)

mathematically, $g(x_0, y_0) < c$.

Then (x_0, y_0) together with $\lambda_0 = 0$ will surely satisfy the K-T.