Homogenous Functions

$F(\vec{x})$ is homogenous if $F(s\vec{x}) = s^k F(\vec{x})$

k = degree of homogeneity

Homothetic Functions

$F(\vec{x})$ is homothetic if $F(\vec{x}) = F(\vec{y}) \Rightarrow F(s\vec{x}) = F(s\vec{y})$

Examples

Cobb-Douglas: $F(K,L) = AK^{\alpha}L^{\beta}$

Homogenous: Let $K = sK,\ L = sL \Rightarrow A(sK)^{\alpha}(sL)^{\beta} = As^{\alpha}K^{\alpha}s^{\beta}L^{\beta}$

$= s^{\alpha+\beta}AK^{\alpha}L^{\beta} = s^{\alpha+\beta}F(K,L)$ //.

Homothetic: Suppose $A\hat{K}^{\alpha}\hat{L}^{\beta} = AK^{\alpha}L^{\beta}$

$\Rightarrow A(s\hat{K})^{\alpha}(s\hat{L})^{\beta} = A(sK)^{\alpha}(sL)^{\beta}$?

$\Rightarrow \cancel{s^{\alpha+\beta}}A\hat{K}^{\alpha}\hat{L}^{\beta} = \cancel{s^{\alpha+\beta}}AK^{\alpha}L^{\beta}$ //.

Homogenous functions are homothetic:

Suppose F is homogenous:

$F(\vec{x}) = F(\vec{y}) \Rightarrow F(s\vec{x}) = s^k F(\vec{x}) = s^k F(\vec{y})$

$= F(s\vec{y})$ //.

Not all homothetic functions are homogenous! Ex. $F(x,y) = xy + 1$

Any strictly increasing transformation of homog. function is homothetic:

$$F(\vec{x}) = H(f(\vec{x}))$$

$f(\vec{x})$ is homogenous

$H(\cdot)$ is strictly increasing

If $F(\vec{x}) = F(\vec{y})$ $\quad H(f(\vec{x})) = H(f(\vec{y}))$

i) $\underline{f(\vec{x}) = f(\vec{y})}$

ii) $F(s\vec{x}) = H(f(s\vec{x})) = H(s^k f(\vec{x})) = H(s^k f(\vec{y}))$

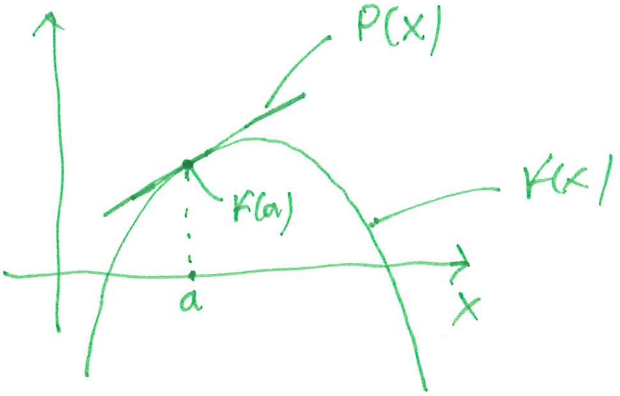
$= H(f(s\vec{y})) = F(s\vec{y})$

Polynomial approximations

Linear approximations

We approximate $f(x)$ near a.

$$f(x) \simeq f(a) + f'(a)(x-a) = P(x)$$



Example

a) $f(x) = \sqrt[3]{x}$ around $a = 1$

$f(1) = \sqrt[3]{1} = 1$ $\qquad f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ $\qquad f'(1) = \frac{1}{3}$

$$\sqrt[3]{x} \simeq P(x) = 1 + \frac{1}{3}(x-1)$$

b) $\underbrace{\ln(1+x)}_{f(x)}$ near $a = 0$

$f(0) = 0$ $\qquad f'(x) = \frac{1}{1+x}$, $f'(0) = 1$

$$P(x) = 0 + 1\cdot(x-0) = x \simeq \ln(1+x)$$

Quadratic approximations

$$f(x) \approx P_2(x) = \underline{A} + \underline{B}(x-a) + \underline{c}(x-a)^2 \qquad \text{near } a$$

$$P'(x) = B + 2c(x-a), \quad P''(x) = 2c$$

$$f(a) = P(a) \quad \Rightarrow \quad f(a) = A$$

$$f'(a) = P'(a) \quad \Rightarrow \quad f'(a) = B$$

$$f''(a) = P''(a) \quad \Rightarrow \quad f''(a) = 2c \iff c = \tfrac{1}{2} f''(a)$$

For x close to a we therefore have:

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2$$

Example: $f(x) = \sqrt[3]{x}$ (close to $a = 1$)

$$f(1) = 1, \quad f'(x) = \tfrac{1}{3} x^{-\frac{2}{3}}, \quad f'(1) = \tfrac{1}{3}, \quad f''(x) = -\tfrac{2}{9} x^{-\frac{5}{3}}$$

$$f''(1) = -\tfrac{2}{9} \cdot 1^{-\frac{5}{3}} = -\tfrac{2}{9}$$

$$f(x) \approx 1 + \tfrac{1}{3}(x-1) + \tfrac{1}{2}\cdot\left(-\tfrac{2}{9}\right)(x-1)^2$$

$$= \underbrace{\underbrace{1 + \tfrac{1}{3}(x-1)}_{\text{linear approx.}} - \tfrac{1}{9}(x-1)^2}_{\text{Quadratic approx.}}$$

Another quadratic approx. example

$y = y(x)$ near $x=0$ when y is implicitly defined near $(0,1) = (x,y)$ given by:

$$xy^3 + 1 = y \qquad y = y(x)$$

$$\frac{dy^2}{dx} = 2yy'$$

$$y' = y^3 + 3xy^2y'$$

Subst. $x=0$, $y=1$ $\quad y' = 1^3 + 3\cdot 0 \cdot 1^2 \cdot y' = 1$

$$y'' = 3y^2y' + 3y^2y' + \cancel{8}\, 6xy(y')^2 + 3xy^2y''$$

Subst. $x=0$, $y=1$

$$y'' = 3\cdot 1^2 \cdot y' + 3 \cdot 1^2 \cdot y' + 6\cdot 0\cdot 1\cdot y'^2 + 3\cdot 0\cdot 1^2 \cdot y''$$

$$= 6y' = 6$$

Thus $y(x)$ near $x=0$

$$y(x) \approx y(0) + y'(0)x + \frac{1}{2}y''(0)x^2 = 1 + x + 3x^2$$

Higher order approximations

Why stop with quadratic approximations? No reason.

$$f(x) \approx P_n(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots + A_n(x-a)^n$$

$$f(a) = P_n(a) \quad \Rightarrow \quad A_0 = f(a)$$

$$f^{(i)}(a) = P_n^{(i)}(a) \qquad \text{for } i = 1, 2, \dots, n$$

$$\Rightarrow A_i = \frac{1}{i!} f^{(i)}(a) \qquad i! = 1 \cdot 2 \cdot 3 \cdots i$$

Gives the Taylor approximation near a!

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

nth order Taylor polynomial.

If $f(x)$ is a polynomial with a degree less than n, the approximation is exact.

Examples

a) e^x near $x=0$ ($a=0$) $\qquad \frac{d}{dx} e^x = e^x \qquad e^0 = 1$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

b) $\ln(1+x)$ near $x=0$ $(a=0)$ $\qquad \frac{d}{dx} = \frac{1}{1+x}\Big|_{x=0} = 1$

$\ln(1) = 0 = f(a)$ $\qquad \frac{d^2}{dx^2} = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1$

$$\ln(1+x) \approx x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

Taylor's formula

For x close to 0 $(a=0)$:

$$f(x) \simeq f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{1}{n!}f^{(n)}(0)x^n + \underbrace{R_{n+1}(x)}_{\text{remainder}}$$

We have that

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)\, x^{n+1} \quad \text{for some } z \in [0, x]$$

This provides an estimate for the error in the approximation:

If $\left| f^{(n+1)}(x) \right| \leq M$ for an interval I

$$\Rightarrow \left| R_{n+1}(x) \right| \leq \frac{M}{(n+1)!} |x|^{n+1}$$

If M is finite, a large n or a small x $\Rightarrow$ small error

Suppose $f(x)$ is differentiable $n+1$ times in an interval that includes 0 and x. Then

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{1}{(n+1)!}f^{(n+1)}(z)\, x^{n+1}.$$

Remainder, examples

Remainder: $R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)\, x^{n+1}$

Linear approximation of $f(x) = \sqrt{25+x}$ near $x=0$

Estimate $\sqrt{25.01}$ with a bound for the remainder.

$$f(x) \simeq \underbrace{f(0)}_{5} + \underbrace{f'(0)}_{\frac{1}{2}(25+x)^{-\frac{1}{2}}\big|_{x=0}} x = 5 + \frac{1}{10}x$$

$$f(0.01) \simeq 5 + \frac{1}{10}\cdot 0.01 = 5.001$$

$$R_2(x) = \frac{1}{2} f''(z)\, x^2 = \frac{1}{2}\left(-\frac{1}{4}(25+z)^{-\frac{3}{2}}\right)x^2$$

$$= -\frac{1}{8}(25+z)^{-\frac{3}{2}} x^2$$

$z \in [0, 0.01]$ $\qquad 25 + z \geq 25 \Rightarrow (25+z)^{-\frac{3}{2}} \leq 25^{-\frac{3}{2}} = \frac{1}{125}$

$$|R_2(0.01)| = \left|-\frac{1}{8}(25+z)^{-\frac{3}{2}}(0.01)^2\right| \leq \frac{1}{8}\cdot\frac{1}{125}\cdot 0.0001$$
$$= 10^{-7}$$

for x close to 0.

For x close to a:

$$R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(z)\,(x-a)^{n+1}$$