Taylor approximation

For x close to a:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots + \frac{f^{(n)}}{n!}(x-a)^n$$

$$f(T) = (1 - e^{-rT})\frac{c}{r} \quad \text{(Value of an income stream)}$$

Second order approx: $f(T) \approx f(0) + f'(0)T + \frac{1}{2}f''(0)T^2$
around $x=0$ $(a=0)$

$$f'(T) = e^{-rT}c \qquad f''(T) = -re^{-rT}c$$

$$f(0) = 0 \qquad f'(0) = c \qquad f''(0) = -rc$$

$$f(T) \approx 0 + cT - \frac{1}{2}rcT^2$$

Let $c = 25$, $T = 3$, $r = 0.05$

Linear: $f(3) \approx 25 \cdot 3 = 75$

Quadratic: $f(3) \approx 25 \cdot 3 - \frac{1}{2} \cdot 0.05 \cdot 25 \cdot 3^2 = 69.375$

Remainders

$$R_{n+1}(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} \quad (\text{for } a=0)$$

Linear: $R_2(x) = \frac{1}{2}f''(z)x^2$

$$R_2(3) = \frac{1}{2}(-rce^{-rz}) \cdot 3^2$$

$z \in [0,3]$

$$|R_2(3)| \leq \left|\frac{1}{2}(-r \cdot c \cdot e^{-r \cdot 0}) \cdot 3^2\right| = \frac{1}{2} \cdot 9 \cdot r \cdot c$$

Plug in $c=25$, $r=0.05$ $\Rightarrow$ $|R_2(3)| \leq \frac{1}{2} \cdot 9 \cdot 0.05 \cdot 25 = 5.625$

Quadratic: $R_3(x) = \frac{1}{6} f'''(z) \cdot x^3$

$$R_3(3) = \frac{1}{6} r^2 e^{-rz} \cdot c \cdot 3^3$$

$$|R_3(3)| \leq \left|\left(\frac{1}{6} r^2 \cdot e^{-r \cdot 0} \cdot c \cdot 3^3\right)\right| = \frac{3^3}{6} r^2 \cdot c$$

With $c = 25$, $r = 0.05$ this becomes:

$$|R_3(3)| \leq \frac{3^3}{6} \cdot 0.05^2 \cdot 25 = 0.2819\ldots$$

Review of differential equations

First order: $\dot{x} = f(x,t)$

Separable: $\frac{dx}{dt} = f(t)g(x)$

i) Separate! $\frac{1}{g(x)}dx = f(t)\,dt$

ii) Integrate! $\int \frac{1}{g(x)}dx = \int f(t)\,dt$

iii) Evaluate and solve for x.

Linear! $\dot{x} + a(t)x = b(t)$

Formula! $x(t) = e^{-A(t)}\left(\int b(t)e^{A(t)}dt + C\right)$ $\quad A(t) = \int a(t)\,dt$

$a(t) = a \implies A(t) = at$

General Solution! Class of functions, an arbitrary constant C

Particular solution! Initial value $x(0) = x_0 \implies$ solve for C!

$(t,x) = (0, x_0)$

Trivial solutions (Constant solutions): $x(t) = A$ $\quad$ for all t

e.g. $x(t) = 0$

Integration recap

Indefinite integral: $\int f(x)dx = F(x) + c$ $\qquad \frac{d}{dx} F(x) = f(x)$

Definite integral: $\int_a^b f(x)dx = F(b) - F(a)$

Integration by parts: $\int fg' = fg - \underline{\int f'g}$

Integration by substitution: find $u = g(x)$, get rid of x, $\quad du = g'(x)dx$

integrate $\int u\,du$

Exam 8.12.2014

Problem 2

a) $\int \frac{1}{x \ln|x|} dx = \ln|\ln|x|| + c$

$\int \frac{1}{u} du$?

Show by substitution:

$u = \ln|x| \quad du = \frac{1}{x} dx$

$\int \frac{1}{\ln|x|} \frac{1}{x} dx = \int \frac{1}{u} du = \ln|u| \overset{+c}{=} \ln|\ln|x|| + c$

b) $\frac{dx}{dt} = (x \ln x)(1 + \ln t) \qquad t \geq 1, \ x \geq 1$

$\Rightarrow \underbrace{\int \frac{1}{x \ln x} dx}_{} = \int (1 + \ln t) dt$

$\int \ln t \, dt \overset{\text{Int. by Parts}}{=} t \ln t - \int t \frac{1}{t} dt = t \ln t - t + c$

$\ln(\ln(x)) = t + t \ln t - t + c$

$\ln(\ln(x)) = t \ln t + c$

$\Rightarrow \quad \ln(x) = e^{t \ln t + c} = k e^{\ln(t^t)} = k t^t$

$\Rightarrow \quad \underline{x(t) = e^{k t^t}}$ k is constant, general solution.

c) Particular solution when $(t, x) = (1, 1) \Rightarrow x(1) = 1$

$x(1) = e^{k \cdot 1^1} = 1 \quad \Leftrightarrow \quad e^k = 1$

$\underline{k \Rightarrow 0}$

$x(t) = e^{0 \cdot t^t} = 1$ for all t, Particular solution.

Exam 4.6.2013

Problem 2: $F(t) = t^t - \frac{1}{\sqrt[3]{3}} = t^t - 3^{-\frac{1}{3}}$

a) show $\int t^t (1 + \ln t)\, dt = F(t) + C$

$$\frac{dF}{dt} = \frac{\partial}{\partial t} t^t = (1 + \ln t) e^{t \ln t} = \underline{t^t (\ln t + 1)}$$

$$\frac{\partial}{\partial k} t^k = \frac{\partial}{\partial k} e^{k \ln t} = \ln t \, e^{k \ln t} = \ln t \, t^k$$

$$\frac{dF}{dt} = \frac{\partial}{\partial t} e^{t \ln t} = $$

$$= e^{t \ln t} \frac{\partial}{\partial t}(t \ln t) = e^{t \ln t}\left(\ln t + t \frac{1}{t}\right)$$

$$= e^{t \ln t} (\ln t + 1)$$

$$\int_0^{\ln 2} (z+1) e^{z(1+e^z)}\, dz$$

Hint: use $t = e^z \Leftrightarrow z = \ln t$

$dt = e^z dz$

$$\Rightarrow \int \underbrace{(z+1)}_{\ln t + 1} \underbrace{e^{z e^z}}_{t^t} \cdot \underbrace{e^z dz}_{dt} = \int (\ln t + 1) t^t \, dt$$

$$= t^t + C = e^{z e^z} + C$$

$$\int_0^{\ln 2} (z+1) e^{z(1+e^z)}\, dz = \Big|_0^{\ln 2} e^{z e^z} = e^{\ln 2 \cdot e^{\ln 2}} - e^{0 \cdot e^0} = 2^2 - 1$$

$$= 3$$

Show by antidifferentiation

8.12.2014

Problem 2

(a) Use integration by substitution to show that $\int \frac{1}{x \ln|x|}\, dx = \ln\big|\ln|x|\big| + C$.
(Integration by substitution is mandatory. There is no score for differentiating the right-hand side.)

(b) Find the general solution of the differential equation

$$\dot{x} = (x \ln x)(1 + \ln t), \qquad t \geq 1,\ x \geq 1 \qquad \text{(D)}$$

(c) Find the particular solution which passes through the point $(t, x) = (1, 1)$.

4.6.2013

Problem 2 Consider the function $F(t) = t^t - \frac{1}{\sqrt[3]{3}}$ defined for $t > 0$.

(a) Show that

$$\int \underbrace{t^t(1 + \ln t)}_{F'(t)}\, dt = F(t) + C$$

and use this to find

$$\int_0^{\ln 2} (z+1)e^{z(1+e^z)}\, dz.$$

(*Hint:* for the latter, use the substitution $t = e^z$.)

(b) How many zeroes does F have?
(You are not asked to calculate any, but observe that $F(1/3) = 0$ and that $F'(1/3) < 0$.)

(c) Use part (a) to find the particular solution which passes through $(t_0, x_0) = (2, 2)$ of the differential equation

$$3\dot{x}(t) = \frac{t^t(1 + \ln t)}{(x(t))^2}$$

10.12.2013

Problem 2 Let $p > 0$ be a constant and let $g(x) = (2x + p)e^{-px} + pxe^{-2px}$.

(a) (i) Find $\lim_{x \to -\infty} g(x)$ or show that it does not exist.

(ii) Show that $\lim_{x \to +\infty} g(x) = 0$.

(b) Show that there exists an $\hat{x} < 0$ such that $g(\hat{x}) = 0$.

(c) Find $\int g(x)\, dx$ and decide whether $\int_p^{\infty} g(x)\, dx$ converges.

b) $F'(t) < 0$ for $\underbrace{(0, \frac{1}{e})}_{\text{at most one zero}}$, $F'(t) > 0$ $\underbrace{(\frac{1}{e}, \infty)}_{\text{at most one}}$

Note: at $t = \frac{1}{3} \Leftrightarrow F(\frac{1}{3}) = 0$

$\frac{1}{3} < \frac{1}{e}$

$\Rightarrow \underline{F(\frac{1}{e}) < 0}$

$\lim_{t \to \infty} F(t) = \infty$

$\Rightarrow$ Positive at some t point for $(\frac{1}{e}, \infty)$

$\Rightarrow$ <u>two zeros</u>

(c)

$$3\dot{x} = \frac{t^t(1+\ln t)}{x^2} \qquad (t_0, x_0) = (2, 2)$$

$$\Rightarrow \quad x(2) = 2$$

$$3\frac{dx}{dt} = \frac{t^t(1+\ln t)}{x^2}$$

$$\Leftrightarrow \quad 3x^2\,dx = t^t(1+\ln t)\,dt$$

$$\int 3x^2\,dx = \int t^t(1+\ln t)\,dt$$

$$x^3 = t^t + C \quad \Rightarrow \quad x(t) = \sqrt[3]{t^t + C}$$

$$x(2) = 2 \Leftrightarrow \quad x(2)^3 = 2^3 = 2^2 + C$$

$$C = 2^3 - 2^2 = 4$$

$$\Rightarrow \quad x(t) = \sqrt[3]{t^t + 4} \qquad \text{Particular solution}$$

$$2 = \sqrt[3]{t^t + c}$$

Exam [illegible] 10.12.2013

Problem 2: $\underline{p>0}$ $g(x) = (2x+p)e^{-px} + pxe^{-2px}$

a) i) Find $\lim\limits_{x\to-\infty} g(x) = -\infty\cdot\infty + (-\infty)\cdot\infty$

$\Rightarrow$ does not converge!

ii) Find $\lim\limits_{x\to\infty} g(x) =$ "$\infty \cdot 0$" + "$\infty\cdot 0$"

Polynomial growth, exponential decay

$\lim\limits_{x\to\infty} xe^{-x} = 0$ $\qquad = 0$

b) As $x \to -\infty$ then $g(x) \to -\infty$

E.g. $g(0) > 0$, Intermediate value theorem (by continuity)

c) Find $\int g(x)dx = \int \left((2x+p)e^{-px} + pxe^{-2px}\right)dx$

$$= \int \underbrace{2xe^{-px}}_{\text{Int. by parts.}} dx + \underbrace{\int pe^{-px}dx}_{-e^{-px}} + \int \underbrace{pxe^{-2px}}_{\text{Int. by parts.}} dx$$

$$\int \underbrace{2x}_{f}\underbrace{e^{-px}}_{g'} dx = \overset{f}{2x}\overbrace{\left(-\frac{1}{p}e^{-px}\right)}^{g} - \int \underbrace{2}_{f'=2}\underbrace{\frac{1}{p}e^{-px}}_{g} dx$$

$$= -2\frac{x}{p}e^{-px} - \frac{2}{p^2}e^{-px}$$

$$\int pxe^{-2px}dx \overset{\text{Int. by parts}}{=} -\frac{1}{2}xe^{-2px} - \frac{1}{4p}e^{-2px}$$

$$\int g(x)\,dx = C \underbrace{- e^{-px} - \frac{2}{p^2} e^{-px}(px+1) - \frac{1}{4p} e^{-2px}(2px+1)}_{G(x)}$$

Does $\int_p^\infty g(x)\,dx$ converge?

$$\int_p^\infty g(x)\,dx = \lim_{b\to\infty} \int_p^b g(x)\,dx = \lim_{b\to\infty} \left(G(b) - G(p)\right)$$

$G(p)$ finite $\Rightarrow$

$$\lim_{b\to\infty} \left(- e^{-px} - \frac{2}{p^2} e^{-px}(px+1) - \frac{1}{4p} e^{-2px}(2px+1)\right)$$

Exponential decay, polynomial growth $\Rightarrow$ limit equals 0.