Differential equations

Suppose we know that:

$$\dot{x}(t) = f(x,t) \qquad (i) \qquad\qquad (\dot{x}(t) = x'(t))$$

What is the function $\underline{x(t)}$?

Equations like (i) are called differential equations.

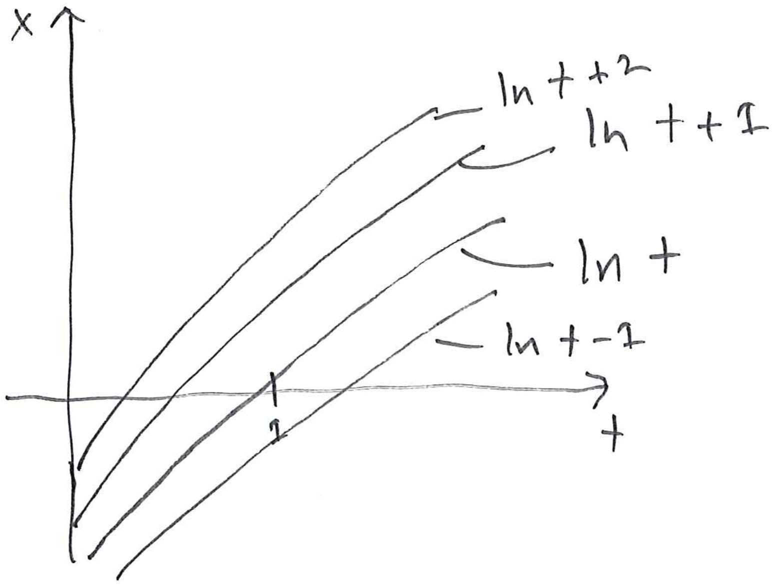
Differential equations describe a class of functions (family of curves)

The simplest case when $f(x,t) = f(t)$. E.g.

$$x'(t) = \frac{1}{t} \quad , \quad t > 0$$

The family of curves this equation describes is

$$x(t) = \ln t + C$$



We need an initial value to pin down the right function.

$$x(\hat{t}) = \hat{x} \quad \text{or} \quad x(0) = 0$$

$$\dot{x}(t) = f(x,t), \quad x(0) = x_0$$

Motivational example

$K(t)$ is capital at time t

Suppose it accrues interest at rate $r>0$

Let capital at time 0 be $K(0)=K_0>0$

What is capital at time t?

$$\dot{K}(t) = rK(t) \qquad \text{(interest accrued at time } t\text{)}$$

Let's write this as

$$\frac{dK}{dt} = rK \iff \frac{1}{K}\,dK = r\,dt \qquad \text{(separation of variables)}$$

Integrate both sides!

$$\ln K = rt + C$$

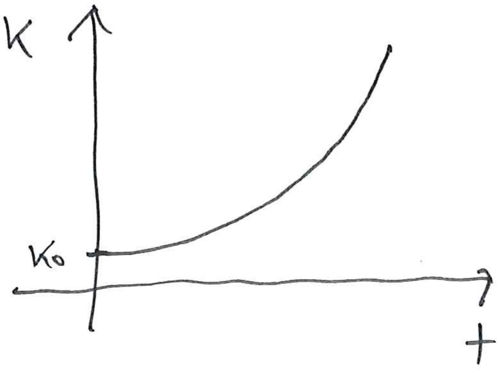
$$e^{\ln K} = K = e^{rt+C} = e^{rt}\underbrace{e^{C}}_{A} = Ae^{rt}$$

$$\Rightarrow K(t) = \underline{\underline{A}}\,e^{rt}$$

$$K(0) = K_0 = Ae^{r\cdot 0} \iff K_0 = A$$

$$\Rightarrow K(t) = K_0 e^{rt}$$

The equation $\dot{x} = rx$ is known as the law of natural growth. It pops up everywhere where growth is present: population growth, GDP growth, and so on.



Terminology

- Ordinary differential equations: one-dimensional diff. eqs. ODE
  i.e. $\dot{x}(t) = f(x, t)$ This course

- First order differential equations, Second order, ...
  → describes what is the highest order derivative.

- Separable differential equations: you separate x and t

- Linear differential equations: linear in the unknown function x

- Trivial solution: $\dot{x} = 0$, $x(t) = 0$

- General solution: the set of all solutions
  For example, $\dot{x} = \frac{1}{t}$ has general solution $x = \ln t + C$

- Particular solution: one function that satisfies the equation

  $\dot{x} = 4$ $\quad x(13) = 42$

  The particular solution is $x = 4t - 10$

  General solution: $x = 4t + C$

  $x(13) = 42 = 4 \cdot 13 + C$

  $C = 42 - 4 \cdot 13 = -10$

- This course: if $x(\hat{t}) = \hat{x}$ given, then there is only one particular solution.

- Initial values: $x(13) = 42 \iff x$ passes through $(t, x) = (13, 42)$

Separable differential equations: general case

An equation of the type

$$\dot{x}(t) = f(t)g(x) \Leftrightarrow \frac{dx}{dt} = f(t)g(x)$$

is called separable. Ex. $\dot{x} = tx$ is separable

$\dot{x} = t + x$ IS NOT

How to solve separable diff. equations:

i) Separate! $\frac{1}{g(x)}dx = f(t)dt$

ii) Integrate both sides!

$$\int \frac{1}{g(x)}dx = \int f(t)dt$$

iii) Evaluate and solve for x.

Example: $\frac{dx}{dt} = e^t x^2$

i) $\frac{1}{x^2}dx = e^t dt$

ii) $\int \frac{1}{x^2}dx = \int e^t dt$ $\quad \frac{1}{x^2} = x^{-2}$

iii) $-\frac{1}{x} = e^t + C \Leftrightarrow -1 = x(e^t + c)$

$\Rightarrow x(t) = -\frac{1}{e^t + C}$ $\Rightarrow x(t) = -\frac{1}{e^t - 2}$

Additionally solution passes through $(t, x) = (0, 1) \Rightarrow x(0) = 1$

$x(0) = 1 = -\frac{1}{1+C} \Leftrightarrow 1 + C = -1 \Rightarrow C = -2$

Seperable diff. eqs, another example

$$\dot{x} = \left(x + \frac{1}{x}\right)\left(t + \frac{1}{t}\right) \quad , \quad \underline{t > 0}$$

$$\frac{dx}{dt} = \left(\frac{x^2 + 1}{x}\right)\left(t + \frac{1}{t}\right)$$

$$\frac{1}{\frac{x^2+1}{x}} dx = \left(t + \frac{1}{t}\right)dt \iff \frac{x}{1+x^2} dx = \left(t + \frac{1}{t}\right)dt$$

$$\int \frac{x}{1+x^2} dx = \int \left(t + \frac{1}{t}\right)dt \quad \Bigg| \quad \int \frac{x}{1+x^2} dx = \int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x\,dx}_{\frac{1}{2}du}$$

Let $u = 1 + x^2$

$du = 2x\,dx \iff \frac{1}{2}du = x\,dx$

$$\Rightarrow \int \frac{1}{u}\frac{1}{2}du = \int \left(t + \frac{1}{t}\right)dt$$

$$\frac{1}{2}\ln(u) = \frac{1}{2}t^2 + \ln(t) + C_1$$

$\ln(t^2) = 2\ln(t)$

$$\ln(1+x^2) = t^2 + 2\ln(t) + 2C_1$$

$$e^{\ln(1+x^2)} = 1 + x^2 = e^{t^2 + 2\ln(t) + 2C_1} = e^{t^2} e^{2\ln(t)} \underbrace{e^{2C_1}}_{K}$$

$$\iff 1 + x^2 = e^{t^2} t^2 K$$

$$x^2 = e^{t^2} t^2 K - 1$$

$$x = \pm\sqrt{e^{t^2} t^2 K - 1}\ , \quad \text{we need } K > 0$$

We need $\pm$ here: sign is determined by the initial condition

Say $x(t_0) = x_0$ If $x_0 > 0$ then $x > 0$

                         If $x_0 < 0$ then $x < 0$

Separable equations, more examples

$$\dot{x} = \left(x + \frac{1}{x}\right)\left(t + \frac{1}{t}\right) \qquad t > 0$$

$$\frac{dx}{dt} = \left(\frac{x^2+1}{x}\right)\left(t + \frac{1}{t}\right)$$

$$\frac{1}{\frac{x^2+1}{x}}\, dx = \left(t + \frac{1}{t}\right)dt \quad \Leftrightarrow \quad \frac{x}{1+x^2}\, dx = \left(t + \frac{1}{t}\right)dt$$

$\Rightarrow \quad \int \frac{x}{1+x^2} dx = \int \left(t + \frac{1}{t}\right)dt$, solve LHS with $u = 1+x^2$, $du = 2x\,dx$

$$\int \frac{1}{u}\frac{1}{2} du = \int \left(t + \frac{1}{t}\right)dt$$

$$\frac{1}{2}\ln(u) = \frac{1}{2}t^2 + \ln(t) + C_1$$

$$\ln(1+x^2) = t^2 + 2\ln(t) + C \qquad C = 2C_1$$

$$e^{\ln(1+x^2)} = 1+x^2 = e^{t^2 + 2\ln(t) + C} = e^{t^2} e^{2\ln(t)} e^{C} = e^{t^2} t^2 K$$

K = constant
$K > 0$

$$\Leftrightarrow \quad x^2 = e^{t^2} t^2 K - 1$$

$$x = \pm\sqrt{e^{t^2} t^2 K - 1}, \quad K > 0 \text{ (otherwise no real solutions)}$$

We need $\pm$ here, the sign is determined by the initial condition. Say $x(t_0) = x_0$

If $x_0 > 0$ then $x > 0$,

If $x_0 < 0$ then $x < 0$.

Population growth

Suppose there is a maximum carrying capacity $K > 0$. Population cannot exceed K. The differential equation for growth is

$$\dot{x}(t) = a\left(K - x(t)\right)$$

Define $u(t) = K - x(t)$

$$\boxed{\dot{u}(t) = -\dot{x}(t)}$$ [illegible]

[illegible] $\dot{u}(t) = -au \iff \frac{du}{dt} = -au \iff \frac{1}{u}du = -a\,dt$

$\iff \ln u = -at + C \iff u(t) = Ae^{-at}$

$u(t) = K - x(t) \iff Ae^{-at} = K - x(t)$

$\Rightarrow x(t) = K - Ae^{-at}$

$x(0) = x_0 \qquad x(0) = K - A = x_0 \qquad A = K - x_0$

$\rightarrow x(t) = K - (K - x_0)e^{-at}$

$$\lim_{t \to \infty} x(t) = K$$

Logistic growth

$K > 0$, we have

$$\dot{x}(t) = rx\left(1 - \frac{x}{K}\right) = rx - \frac{x^2}{K}$$

Let $u(t) = -1 + \frac{K}{x}$ $(x > 0)$

$$\dot{u} = -\frac{K}{x^2}\dot{x} = -K\left(\frac{rx\left(1 - \frac{x}{K}\right)}{x^2}\right) = -K\left(\frac{r\left(1 - \frac{x}{K}\right)}{x}\right)$$

$$= -Kr\left(\frac{1}{x} - \frac{1}{K}\right) = -r\left(\frac{K}{x} - 1\right) = -ru$$

$$\dot{u} = -ru \iff \boxed{u(t) = Ae^{-rt}}$$

$$\frac{du}{dt} = -ru \iff \frac{1}{u}du = -rdt$$

$$-1 + \frac{K}{x} = Ae^{-rt} \iff -x + K = xAe^{-rt}$$

$$x(1 + Ae^{-rt}) = K \iff x = \frac{K}{1 + Ae^{-rt}}$$

general solution

Particular solution: $X(0) = x_0$

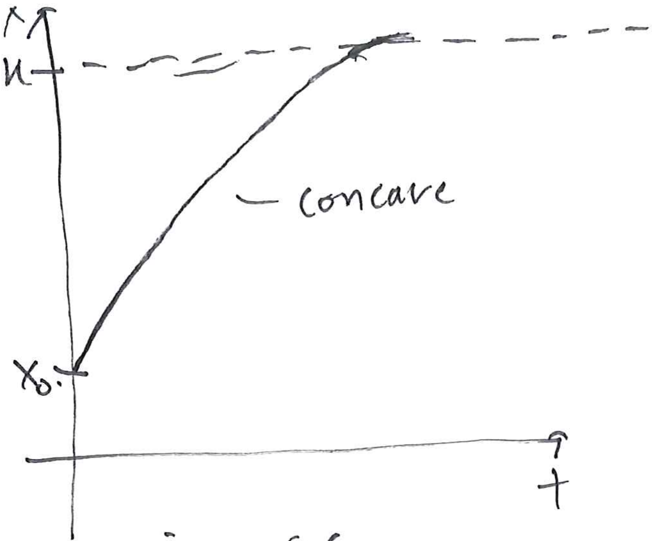
$$x_0 = \frac{K}{1 + A} \iff A = \frac{K}{x_0} - 1 = \frac{K - x_0}{x_0}$$

$$X(t) = \frac{K}{1 + \frac{K - x_0}{x_0}e^{-rt}}$$

$$\dot{x} = a(K - x)$$

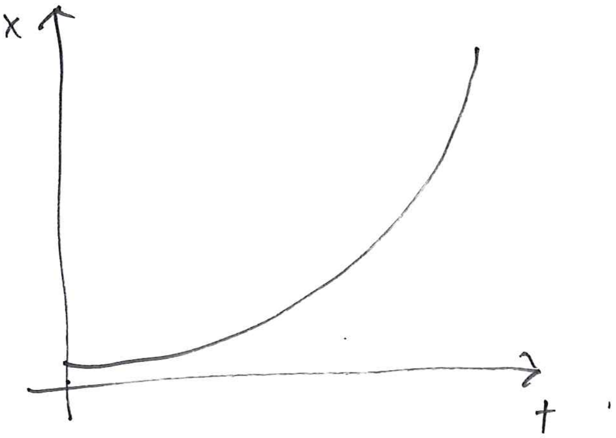
Growth with carrying capacity:

$$x(t) = K - (K - x_0)e^{-at}$$



$$\dot{x} = rx$$

Exponential growth



$$\dot{x} = rx\left(1 - \frac{x}{K}\right)$$

Logistic growth:

$$x(t) = \frac{K x_0}{x_0 + (K - x_0)e^{-rt}}$$