Example ~~of~~ a separable differential equation!

$$\dot{x} = e^{at}\cdot e^{bx}\cdot e^{c} = e^{at+bx+c} \qquad x(0)=1$$

$$\frac{dx}{dt} = e^{at+bx+c} \quad (\Leftrightarrow) \quad e^{-bx}\,dx = e^{at+c}\,dt$$ i) Separate

ii) Integrate

iii) Solve for x

$$\int e^{-bx}\,dx = \int e^{at+c}\,dt$$

$$-\frac{1}{b}e^{-bx} = \frac{1}{a}e^{at+c} + A_1$$

$$e^{-bx} = -\frac{b}{a}e^{at+c} \underbrace{-\,bA_1}_{+A}$$

Take logs!

$$-bx = \ln\left(-\frac{b}{a}e^{at+c} + A\right)$$

$$x = -\frac{1}{b}\ln\left(A - \frac{b}{a}e^{at+c}\right)$$ General solution

$$x(0) = 1 = -\frac{1}{b}\ln\left(A - \frac{b}{a}e^{c}\right)$$

$$-b = \ln\left(A - \frac{b}{a}e^{c}\right) \qquad e^{\ln x} = x$$

$$e^{-b} = A - \frac{b}{a}e^{c}$$

$$A = e^{-b} + \frac{b}{a}e^{c}$$

$$x(t) = -\frac{1}{b}\ln\left(e^{-b} + \frac{b}{a}e^{c} - \frac{b}{a}e^{at+c}\right)$$ Particular solution

Checking the solution!

We say that $\hat{x}(t) = -\frac{1}{b}\ln\left(A - \frac{b}{a}e^{at+c}\right)$

Solves $\dot{x} = e^{at+bx+c} = e^{at+c}e^{bx}$, if and only if

$$\frac{d\hat{x}}{dt} = -\frac{1}{\cancel{b}}\frac{1}{A-\frac{b}{a}e^{at+c}}\left(-\frac{\cancel{b}}{\cancel{a}}\cancel{a}\,e^{at+c}\right) = \frac{e^{at+c}}{A-\frac{b}{a}e^{at+c}}$$

Substitute back to the original equation together with $x(t) = \hat{x}(t)$

$$\Rightarrow \frac{e^{at+c}}{A-\frac{b}{a}e^{at+c}} = e^{at+c}\, e^{\cancel{b}\cdot\left(-\frac{1}{\cancel{b}}\right)\ln\left(A-\frac{b}{a}e^{at+c}\right)}$$

$$= e^{at+c}\, e^{\ln\left(\left(A-\frac{b}{a}e^{at+c}\right)^{-1}\right)} \qquad e^{\ln x} = x$$

$$= e^{at+c}\left(A-\frac{b}{a}e^{at+c}\right)^{-1}$$

$$= \frac{e^{at+c}}{A-\frac{b}{a}e^{at+c}}$$

Linear differential equations (1st order)

Equations that are linear in the unknown function x are called linear differential equations.

$$\underline{\dot{x} + a\underline{x} = b} \quad \text{Linear}, \qquad \dot{x} + a\underline{x^2} = b \quad \text{nonlinear}$$

We focus on first order linear differential equations:

$$\dot{x} + a(t)x(t) = b(t)$$

Two ways to solve these:

i) Use a formula. $x(t) = e^{-A(t)}\left(\int b(t)e^{A(t)}dt + C\right)$

$$A(t) = \int a(t)dt$$

ii) Use integrating factor:

$$\frac{d}{dt}\left(e^{A(t)}x(t)\right) = e^{A(t)}\dot{x} + a(t)e^{A(t)}x(t) = e^{A(t)}\left(\dot{x} + a(t)x(t)\right)$$

Multiply both sides with $e^{A(t)}$

$$\underbrace{e^{A(t)}\left(\dot{x} + a(t)x\right)}_{\frac{d}{dt}(\;'\;)} = e^{A(t)}b(t)$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{A(t)}x(t)\right) = e^{A(t)}b(t)$$

$$\int \left(e^{A(t)}x(t)\right)\frac{d}{dt} = e^{A(t)}x(t) = \int e^{A(t)}b(t)dt + C$$

$$\Rightarrow \quad x(t) = e^{-A(t)}\left(\int e^{A(t)}b(t)dt + C\right)$$

Example $\dot{x} + x = t$ $\quad (\dot{x} + a(t)x = b(t))$

$$x(t) = e^{-A(t)}\left(\int e^{A(t)} b(t)\,dt + c\right)$$ Formula

$b(t) = t$ $\quad A(t) = \int 1\,dt \quad \left(= \int a(t)\,dt\right)$

$= t$

$fg - \int f'g$

$$\Rightarrow x(t) = e^{-t}\left(\int e^{t}\cdot t\,dt + c\right) = e^{-t}\left(e^{t}\cdot t - \int e^{t}\,dt\right) + c)$$

$f'(t) = 1$ $\quad g'(t) = e^{t}$ int. by parts.

$f(t) = t$ $\quad g(t) = e^{t}$

$$= e^{-t}\left(e^{t}\cdot t - e^{t} + c\right) = t - 1 + e^{-t}c$$

Example: $c>0$ $\quad r>0$ $\quad$ up to $t=T$, $X(T)=0$

$$\dot{x} - rx = -c \qquad \Big|\ \text{formula: } X(t) = e^{+rt}\left[\int (-c)e^{-rt} + A\right]$$

$$\frac{d}{dt}\left(e^{-rt}x\right) = -e^{-rt}c$$

$$e^{-rt}x = -\int e^{-rt}c + A \quad \| \cdot e^{rt}$$

$$x = e^{+rt}\left(-\int e^{-rt}c + A\right) = e^{+rt}\left(+\frac{1}{r}e^{-rt}c + A\right)$$

$$= \frac{c}{r} + e^{rt}A \qquad \text{General Solution}$$

$$X(T) = 0 = \frac{c}{r} + e^{+rT}A \iff A = -e^{-rT}\frac{c}{r}$$

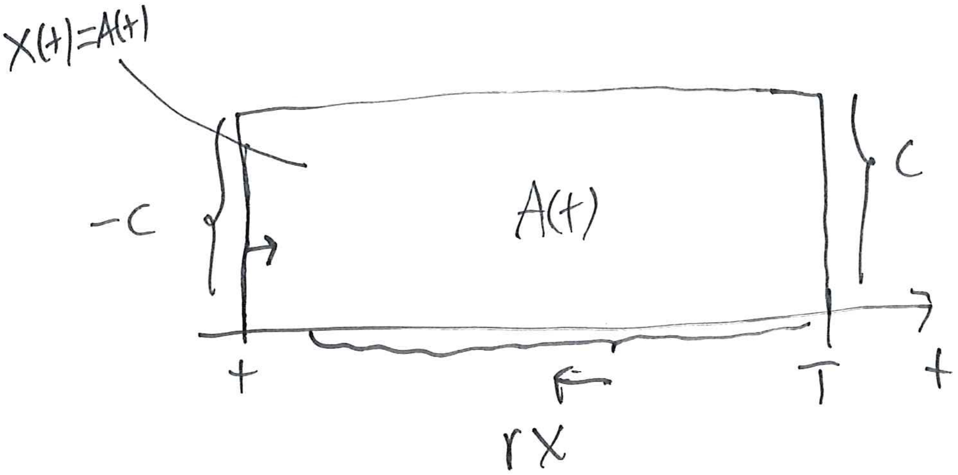
$$A = -e^{-rT}\frac{c}{r}$$

$$\Rightarrow X(t) = \frac{c}{r} + e^{rt}\left(-e^{-rT}\frac{c}{r}\right) = \frac{c}{r} - e^{rt-rT}\frac{c}{r}$$

$$= \left(1 - e^{-r(T-t)}\right)\frac{c}{r} \qquad \text{Particular solution}$$

$$= \int_t^T e^{-rs}c\,ds = \text{"Cash flow from } t \text{ to } T\text{"}$$

$$\dot{x} - rx = -c \iff \dot{x} = rx - c$$



Example from the book:

GDP: $X(t) = 0.2\,K(t)$

Capital $\dot{K}(t) = 0.1\,X(t) + H(t)$

FDI: $H(t) = 10\,e^{0.04t}$

i) Derive differential equation for $K(t)$:

$$\dot{K}(t) = 0.1 \cdot 0.2 \cdot K(t) + 10\,e^{0.04t} = 0.02\,K(t) + 10\,e^{0.04t}$$

ii) Solve it with $K(0) = 200$

$$\dot{K} - 0.02\,K = 10\,e^{0.04t}$$

$A(t) = -0.02t$

$A(t) = \int a(t)\,dt$

Using the formula: $$x(t) = e^{-A(t)}\left(\int b(t)\,e^{A(t)}\,dt + C\right)$$

$$\cancel{= e^{-0.02t}\left(\int 10\,e^{0.04t}\,dt + C\right)}$$

$$= e^{0.02t}\left(\int 10\,e^{0.04t - 0.02t} + C\right) = e^{0.02t}\left(\int 10\,e^{+0.02t}\,dt + C\right)$$

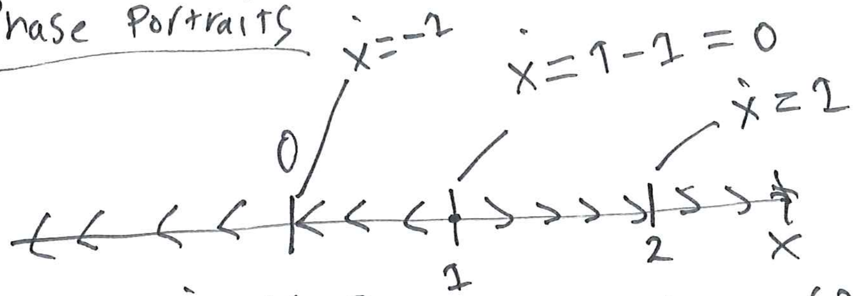
$$= e^{0.02t}\left(\underbrace{\frac{10}{0.02}}_{500}\,e^{0.02t} + C\right) = \underbrace{500\,e^{0.04t} + e^{0.02t}C}_{\text{General solution}}$$

$$K(0) = 200 = 500 + C$$

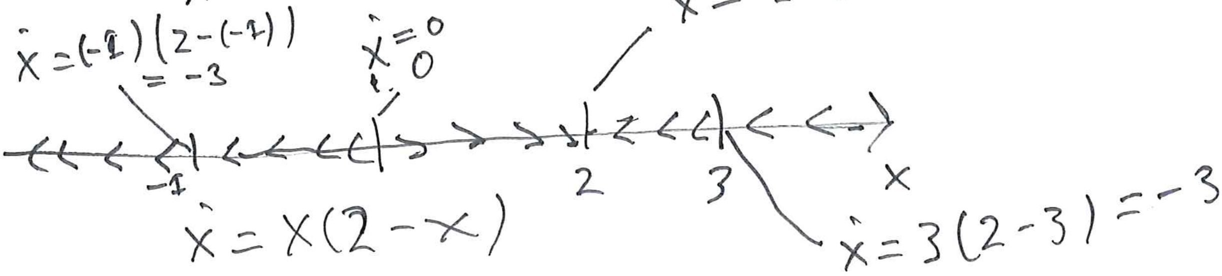
$$C = -300$$

$$K(t) = 500\,e^{0.04t} - 300\,e^{0.02t}$$

Phase Portraits



$\dot{x} = x - 1$

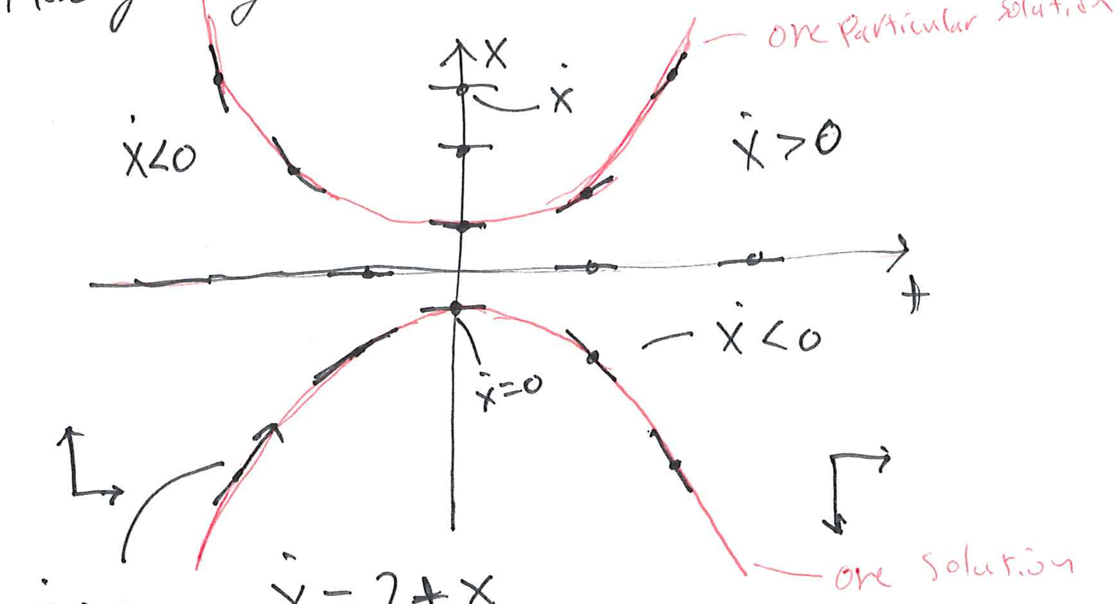


$\dot{x} = x(2 - x)$

This only works if RHS differential equation does not depend on time! $\dot{x} + ax = c$ $\qquad$ $\dot{x} + ax = c(t)$

$\rightarrow$ does not work

More generally we can draw direction fields:



$\dot{x} = 2tx$

$(x = Ke^{t^2})$