Review of differential equations

First order: $\dot{x} = f(x,t)$

Separable: $\frac{dx}{dt} = f(t)g(x)$

i) Separate: $\frac{1}{g(x)}dx = f(t)dt$

ii) Integrate: $\int \frac{1}{g(x)}dx = \int f(t)dt$

iii) Evaluate and solve for x.

Linear: $\dot{x} + a(t)x = b(t)$

Formula: $x(t) = e^{-A(t)}\left(\int b(t)e^{A(t)}dt + C\right)$ $\quad A(t) = \int a(t)dt$

$a(t) = a \Rightarrow A(t) = at$

General Solution: Class of functions, an arbitrary constant C

Particular solution: Initial value $x(0) = x_0 \Rightarrow$ solve for C!

Higher order derivatives

$\ddot{x} + \dot{x} = 0$ (2nd order), Let $y = \dot{x}$

$\dot{y} + y = 0$ (1st order) $x = \int y \, dt$

$\ddot{x} + \dot{x} + \boxed{x} = 0$ => We cannot solve this as a first order equation.

Differenc equations

Time is discrete:

$y_{t+1} = a y_t$, Solution: Suppose $y_0 = B$

$y_1 = aB$, $y_2 = a(aB)$, $y_3 = a(a(aB))$

$y_t = a^t B$ $t \in \{0, 1, 2, 3, 4, \ldots\}$

Systems of equations

We can have a system of equations, example:

$$\begin{cases} \dot{x} + ax + y = 0 \\ \dot{y} + by + x = 0 \end{cases}$$

Integration by parts: definite integrals

Recall: $\int f(x)g'(x)\,dx = \underbrace{f(x)g(x) - \int f'(x)g(x)\,dx}_{F(x)}$

for <u>indefinite</u> integrals.

For definite integrals, we can:

i) Use the above to find the indefinite integral $F(x)$ and then evaluate $F(b) - F(a)$

ii) Use the following:

$$\int_a^b f(x)g'(x)\,dx = \Big|_a^b f(x)g(x) - \int_a^b f'(x)g(x)\,dx$$

Example:

$$\int_0^{10} (1+0.4t)e^{-0.05t}\,dt$$

Choose $f(t) = 1 + 0.4t$ $\quad g'(t) = e^{-0.05t}$

$f'(t) = 0.4$ $\quad g(t) = -\frac{1}{0.05}e^{-0.05t} = -20e^{-0.05t}$

$$\int_0^{10}(1+0.4t)e^{-0.05t}\,dt = \Big|_0^{10}(1+0.4t)\left(-20e^{-0.05t}\right) - \int_0^{10} 0.4\left(-20e^{-0.05t}\right)dt$$

$$= \underbrace{(1+0.4\cdot 10)}_{1+4}\left(-20e^{-0.05\cdot 10}\right) - \underbrace{\left(1(-20)e^{-0.05\cdot 0}\right)}_{20} + 8\int_0^{10} e^{-0.05t}\,dt$$

$$= -5\cdot 20\cdot e^{-0.5} + 20 + 8\Big|_0^{10} -\frac{1}{0.05}e^{-0.05t} = -100e^{-0.5} + 20 - \frac{8}{0.05}\left(e^{-0.5} - 1\right)$$

$$= -100e^{-0.5} + 20 - 160\left(e^{-0.5} - 1\right) = \underline{180 - 260e^{-0.5}} \approx 22{,}3$$

Leibniz rule

$$\frac{d}{db}\left(\int_a^b f(t)\,dt\right) = f(b) \qquad \frac{d}{da}\left(\int_a^b f(t)\,dt\right) = -f(a)$$

Now, let $a = a(x)$, $b = b(x)$, $f = f(x,t)$

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

$$\boxed{F'(x) = f(x, b(x))\,b'(x) - f(x, a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt \qquad \text{Leibniz rule}}$$

Why? $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt = G(x, b(x)) - G(x, a(x))$

$$G(x,t) = \int f(x,t)\,dt$$

Use the Chain rule:

$$\frac{dF}{dx} = \underbrace{\frac{\partial G}{\partial x}}_{\frac{\partial G}{\partial x}(x,b(x))} \frac{\partial G}{\partial b}\frac{\partial b}{\partial x} - \frac{\partial G}{\partial x}(x, a(x)) - \frac{\partial G}{\partial a}\frac{\partial a}{\partial x}$$

$$\frac{dF}{dx} = f(x, b(x))\,b'(x) - f(x, a(x))\,a'(x) + G_1(x, b(x)) - G_1(x, a(x)) \qquad G_1 = \frac{\partial G}{\partial x}$$

$$\int_{a(x)}^{b(x)} f(x,t)\,dt = G(x, b(x)) - G(x, a(x))$$

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,t)\,dt = \frac{\partial}{\partial x}\Big(G(x,b(x)) - G(x,a(x))\Big)$$

$$\Leftrightarrow \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt = G_1(x,b(x)) - G_1(x,a(x))$$

we are done!

Example: $F(x) = \int_{x^2}^{x} t\, e^{te^x} dt$, Show $F'(0) = 0$

$$F'(x) = \underbrace{x e^{xe^x}}_{f(x,x)} \cdot \underbrace{1}_{b'(x)} - \underbrace{x^2 e^{x^2 e^{x^2}}}_{f(x,x^2)} \cdot \underbrace{2x}_{a'(x)} + \int_{x^2}^{x} \underbrace{t^2 e^x \cdot e^{te^x}}_{\frac{\partial}{\partial x} f(x,t)} dt$$

$$\frac{\partial}{\partial x} e^{f(x)} = f'(x) e^{f(x)}$$

$$\int_0^0 g(x,t)\, dt = 0 = G(0) - G(0)$$

When $x = 0$

$$F'(0) = 0 + 0 + 0 = 0$$

Ex. Find $\frac{d}{dx} \int_1^e t^{-1} e^{(1+x^2)t} dt = F'(x)$

$$F'(x) = 0 + 0 + \int_1^e \frac{\partial}{\partial x} \left(t^{-1} e^{(1+x^2)t} \right) dt$$

$$= \int_1^e \cancel{t^{-1}}\, 2x \cancel{t}\, e^{(1+x^2)t} dt = \Big|_1^e 2x \frac{1}{1+x^2} e^{(1+x^2)t} (+C)$$

$$= \frac{2x}{1+x^2} \Big|_1^e e^{(1+x^2)t} = \frac{2x}{1+x^2} \left(e^{(1+x^2)e} - e^{1+x^2} \right)$$

Infinite intervals of integration

Suppose we are integrating from a to ∞. Can we do this?

Is A finite? $A < \infty$?

$$A = \int_a^\infty f(x)\,dx$$

Contrast: $\sum_{i=1}^{\infty} a_i < \infty$?

$a_i = \left(\frac{1}{n^2} = \right) \frac{1}{i^2}$ finite

$a_i = \frac{1}{i}$ infinite

Formally, we write the integral from a to ∞ as

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \int_a^\infty f(x)\,dx \qquad (f(x) \text{ integrable})$$

The question is: does the integral converge? E.g.

$$\lim_{b\to\infty} \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \underbrace{\lim_{b\to\infty} \int_c^b f(x)\,dx}_{<\varepsilon?} \qquad (a < c) \qquad \varepsilon > 0$$

If the limit does not exist, we say the integral diverges

$$\left(\lim_{b\to\infty} \int_a^b f(x)\,dx = \pm\infty\right)$$

Similarly for $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

Example. a) $\int_0^\infty \lambda e^{-\lambda x}\,dx \quad , \lambda > 0$

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda x}\,dx \quad \Big| \int \lambda e^{-\lambda x}\,dx = -e^{-\lambda x} + C$$

$$= \lim_{b\to\infty}\left(\Big|_0^b \left(-e^{-\lambda x} + C\right) \right) = \lim_{b\to\infty}\left(-e^{-\lambda b} + \underbrace{e^{-\lambda \cdot 0}}_{=1}\right)$$

$$= \lim_{b\to\infty}\left(1 - e^{-\lambda b}\right) = 1$$

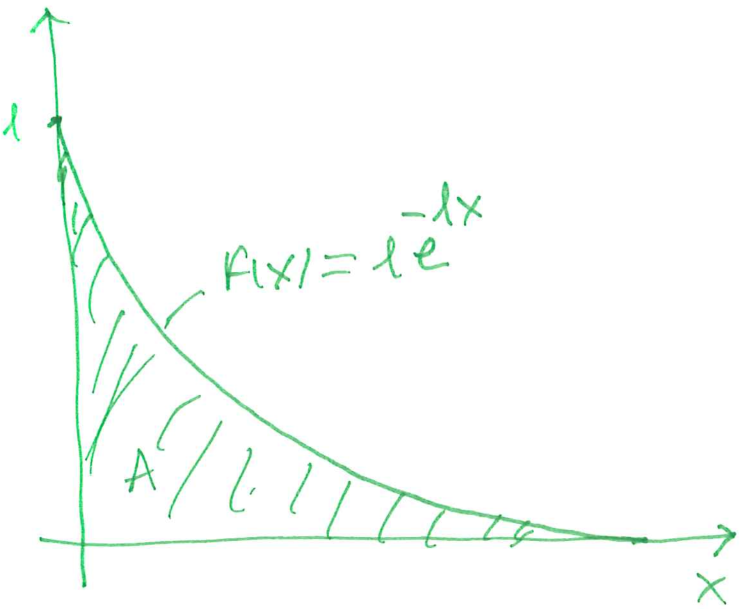
Converges

b) $$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \Big|_1^b \left(\ln|x|\right)$$

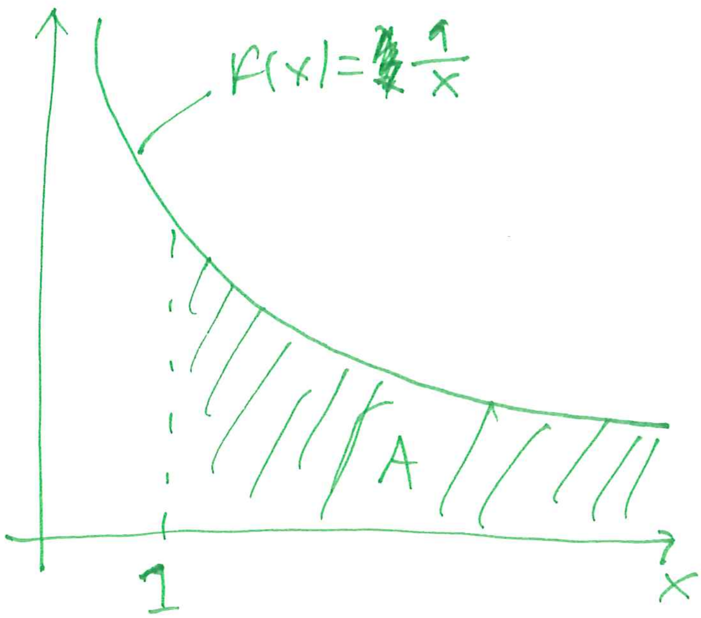
$$= \lim_{b\to\infty}\left(\ln(b) - \underbrace{\ln(1)}_{=0}\right) = \lim_{b\to\infty} \ln(b) = \infty$$

Diverges!

a/



f(x) approaches 0 rapidly



f(x) approaches 0 slowly