Example of divergence

$$A = \int_a^{\infty} e^{ax}\,dx = \infty$$

$$= \lim_{b \to \infty} \int_a^b e^{ax}\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \lim_{a \to -\infty} \int_a^0 f(x)\,dx + \lim_{b \to \infty} \int_0^b f(x)\,dx$$

$$\text{Ex. } \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \underbrace{\int_0^{\infty} \frac{x}{1+x^2}\,dx}_{\text{first do this}} + \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx$$

Use substitution: $u = 1 + x^2$ $\quad du = 2x\,dx \Leftrightarrow \frac{1}{2}du = x\,dx$

$$\Rightarrow \int_0^{\infty} \frac{1}{2u}\,du = \lim_{b \to \infty} \int_0^b \frac{1}{2}\frac{1}{u}\,du = \lim_{b \to \infty} \frac{1}{2} \Big|_0^b \ln u = \lim_{b \to \infty} \frac{1}{2} \Big|_0^b \ln(1+x^2)$$

$$= \lim_{b \to \infty} \frac{1}{2}\Big(\ln(1+b^2) - \underbrace{\ln(1)}_{=0}\Big) = \infty$$

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \infty + \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx \rightarrow \text{diverges!}$$

Integrals of unbounded functions

Consider $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ where $x \in (0, 2]$

$\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$ (not defined at $x=0$)

We can let $\int_0^2 \frac{1}{\sqrt{x}}\,dx = \lim_{h\to 0^+} \int_h^2 \frac{1}{\sqrt{x}}\,dx = \lim_{h\to 0^+} \Big|_h^2 2\sqrt{x}$

$= \lim_{h\to 0^+} \left(2\sqrt{2} - 2\sqrt{h}\right) = 2\sqrt{2} = 2^{1.5}$ Converges!

More generally: Let $f(x)$ be continous on $(a, b]$ (not defined at a)

Then $\int_a^b f(x)\,dx = \lim_{h\to 0^+} \int_{a+h}^b f(x)\,dx$

If this limit exists, the integral converges.

The case when $f(x)$ is cont. on $[a, b)$ (not defined at b)
is analogous:

$$\int_a^b f(x)\,dx = \lim_{h\to 0^+} \int_a^{b-h} f(x)\,dx$$

Finally when $f(x)$ is cont. on (a, b) (not defined at $x=a$, or $x=b$)

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\text{Limit}} + \underbrace{\int_c^b f(x)\,dx}_{\text{Limit}} \qquad a < c < b$$

If both limits exist, the integral converges.

Homogenous functions of two variables

$F(x,y)$ is said to be homogenous function of degree K if, for (x,y) in its domain:

$$F(sx, sy) = s^K F(x,y) \qquad \text{for all } s > 0$$

K can be any number: positive, negative, zero

Economics application: returns to scale

$$F(sx) = s^K F(x) \qquad (s > 1)$$

$K < 1$ decreasing returns to scale

$$s > s^K$$

$K = 1$ constant returns to scale

$$s = s^K$$

$K > 1$ increasing returns to scale

$$s < s^K$$

Euler's theorem:

$F(x,y)$ is homogenous of degree K if and only if

$$x F_1(x,y) + y F_2(x,y) = K F(x,y)$$

$$F_1(x,y) = \frac{\partial F}{\partial x}$$

$$F_2 = \frac{\partial F}{\partial y}$$

Proof "$\Rightarrow$" $F(x,y)$ is homogenous of degree K:

$$F(sx, sy) = s^K F(x,y)$$

$$\frac{d}{ds} F(sx, sy) = \frac{d}{ds} s^K (F(x,y))$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} = K s^{K-1} F(x,y)$$

$$F_1(x,y) x + F_2(x,y) y = K s^{K-1} F(x,y)$$

This holds for all $s>0$, pick $s=1$

$\Rightarrow x f_1(x,y) + y f_2(x,y) = k f(x,y)$.

Properties: $f(x,y)$ is homogenous of degree k

- $f_1(x,y)$, $f_2(x,y)$ are homogenous of degree $k-1$

Proof:
$$\frac{d}{dx} f(sx,sy) = \frac{d}{dx} s^k f(x,y)$$
$$s f_1(sx,sy) = s^k f_1(x,y)$$
$$f_1(sx,sy) = s^{k-1} f_1(x,y) \quad \square$$

Same for $f_2(x,y)$.

- Assume $f(x,y)$ is twice cont. differentiable, then
$$x^2 f_{11}(x,y) + 2xy f_{12}(x,y) + y^2 f_{22}(x,y) = k(k-1) f(x,y)$$

- $f(x,y) = x^k f(1,\frac{y}{x}) = y^k f(\frac{x}{y},1)$ Provided that $x>0, y>0$

Proof: Let $s=\frac{1}{x}$
$$f(sx,sy) = s^k f(x,y)$$
$$f(1,\frac{y}{x}) = \frac{1}{x^k} f(x,y)$$
$$\Rightarrow \quad x^k f(1,\frac{y}{x}) = f(x,y)$$

Examples

Plug in $x=sx$, $y=sy$

a) $\frac{xy}{x^2+y^2}$, Homogenous of degree 0: $\frac{sxsy}{(sx)^2+(sy)^2} = \frac{s^2xy}{s^2x^2+s^2y^2} = \frac{s^2}{s^2}\frac{xy}{x^2+y^2}$
$$= \frac{xy}{x^2+y^2}$$

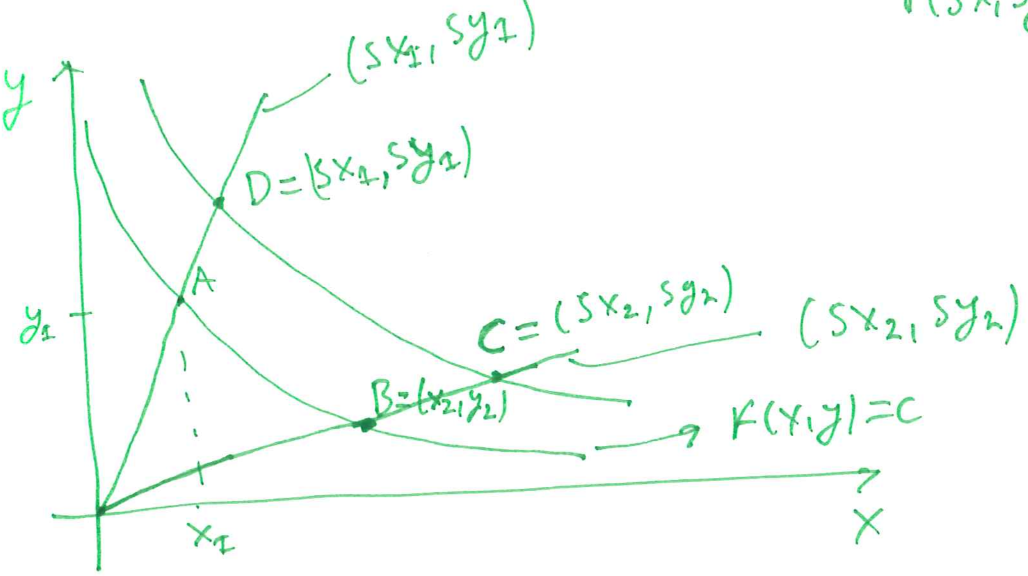
b) xy Homogenous of degree 2: $sxsy = s^{2}xy$ $\quad k=2$

c) $\frac{x^3}{y^2}$, degree 1: $\frac{s^3x^3}{s^2y^2} = s\frac{x^3}{y^2}$

Geometric interpretation of homogeneity

If one level curve of a homogenous function is known, then all of its level curves are known. $f(x,y)=c$

$f(sx,sy) = s^k c$



Pick a point $A=(x_1,y_1)$ on the level curve

i) Draw a ray through A from origin. It gives all points of the form (sx_1, sy_1).

ii) Draw a new ray from the origin through some point $B(x_2,y_2)$ which lies on $f(x,y)=c$.

iii) Pick $s>0$. $D=(sx_1,sy_1)$ and $C=(sx_2,sy_2)$ must level curve, because

$$\underbrace{f(sx_2,sy_2)}_{C} = s^k f(x_2,y_2) = s^k c = s^k f(x_1,y_1)$$

$$= f(\underbrace{sx_1, sy_1}_{D})$$

iv) Repeat, we can find all the points on the level curve.

Tangents to the level curves are ~~parallel~~ Parallel

$F(x,y)$ homogenous of degree k

Implicit function theorem

Level curve $F(x,y)=C$, slope $-\frac{F_1(x,y)}{F_2(x,y)}$ at (x,y)

At point D the slope is:

$$-\frac{F_1(sx_1, sy_1)}{F_2(sx_1, sy_1)} = -\frac{s^{k-1}F_1(x_1 y_1)}{s^{k-1}F_2(x_1 y_1)} = -\frac{F_1(x_1, y_1)}{F_2(x_1, y_1)}$$

We chose x_1, y_1 as our starting point, we could as well choose (a,b)!

Homogenous functions of multiple variables

Suppose f is a function of n variables defined in domain D. D is a cone if whenever $(x_1, x_2, \dots, x_n) \in D$ then $(sx_1, sx_2, \dots, sx_n) \in D$. When D is a cone we say f is homogenous of degree k on D, if

$$f(sx_1, sx_2, \dots, sx_n) = s^k f(x_1, \dots, x_n)$$

for all $s > 0$.

Euler's theorem

$$\sum_{i=1}^{n} x_i f_i(\vec{x}) = k f(\vec{x}) \qquad \vec{x} = (x_1, x_2, \dots, x_n)$$

Proof: Along similar lines to the two-dimensional case.

Economic applications: returns to scale

If we have $F(s\vec{x}) = sF(\vec{x})$, Homogenous of degree 1, we have constant returns to scale.

$s^k F(\vec{x})$ $k<1$ decreasing returns to scale
$k>1$ Increasing returns to scale.

Tangents to the level curves are parallel

$f(x,y)$ is homogenous of degree k

Level curve $f(x,y)=c$, Slope $-\dfrac{f_1(x,y)}{f_2(x,y)} = y'(x)$

At point D the slope equals:

$$-\frac{f_1(sx_1, sy_1)}{f_2(sx_1, sy_1)} = -\frac{s^{k-1} f_1(x_1, y_1)}{s^{k-1} f_2(x_1, y_1)} = -\frac{f_1(x_1, y_1)}{f_2(x_1, y_1)}$$

Homogenous functions of multiple variables

Let f be a function of $\underline{n}$ variables, defined in a cone D. D is a cone if whenever $(x_1, \dots, x_n) \in D$ then $(sx_1, \dots, sx_n) \in D$. When D is a cone we say f is homogenous of degree k on D, if

$$f(sx_1, \dots, sx_n) = s^k f(x_1, \dots, x_n)$$

for all $s > 0$. $\left(f(s\vec{x}) = s^k f(\vec{x}) \right)$ $\vec{x} \in D$

Euler's theorem:

$$\sum_{i=1}^{n} x_i f_i(\vec{x}) = k f(\vec{x})$$

Proof: Similar to two dimensions.

Homothetic Functions

Let f be a function of n variables defined on a cone D.

f is homothetic if

$$f(\vec{x}) = f(\vec{y}) \Rightarrow f(s\vec{x}) = f(s\vec{y}).$$

Interpretation: if consumer is indifferent between two bundles $\vec{x}$ and $\vec{y}$, then she is also indifferent between $s\vec{x}$ and $s\vec{y}$.

Homogenous functions are homothetic:

$$f(\vec{x}) = f(\vec{y}) \Rightarrow f(s\vec{x}) = s^k f(\vec{x}) = s^k f(\vec{y}) = f(s\vec{y}). \quad \square$$