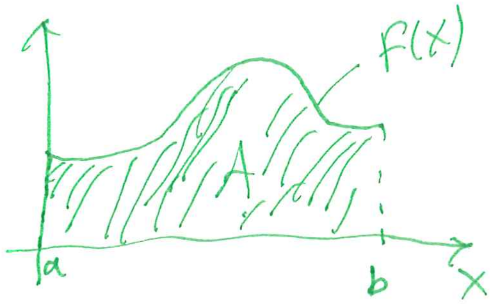
Integration

Motivation

i) Area under a curve

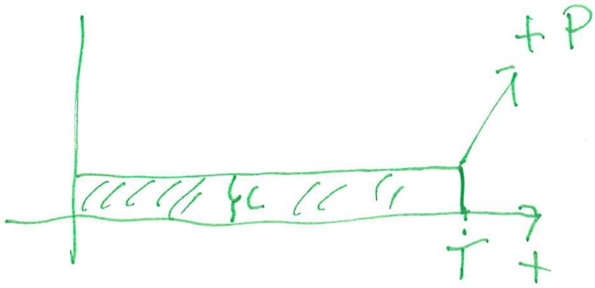


A?

$$A = \int_a^b f(x)\,dx$$

ii) Value of bond: $c > 0$, T, $r > 0$, $P > 0$

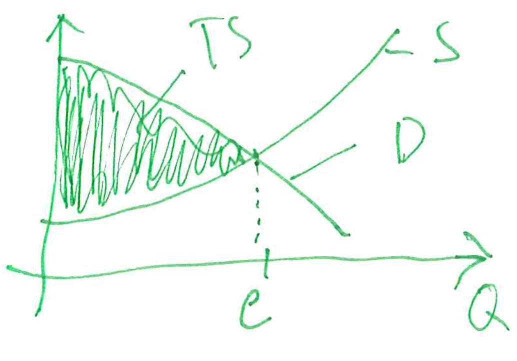
$$V = \int_0^T e^{-rt} c\,dt + e^{-rT} P$$



iii) Probabilities

$$P(x \le x^*) = \int_{-\infty}^{x^*} f(x)\,dx$$

iv) Total Surplus



$$TS = \int_0^e (D - S)\,dQ$$

Indefinite integral (antiderivative)

Derivative:

$$\frac{d}{dx} x^{a+1} = (a+1)x^a$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Antiderivative:

$$\int (a+1)x^a dx = x^{a+1} + C$$

$$\int a e^{ax} dx = e^{ax} + C$$

$$\int \frac{1}{x} dx = \ln x + C$$

Can we do the reverse operation? Can we find the antiderivative?

$$\int x^{a+1} dx = \frac{1}{a+2} x^{a+2} + C \qquad \frac{d}{dx}\left(\frac{1}{a+2} x^{a+2} + C\right) = x^{a+1}$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C \qquad \frac{d}{dx}\left(\frac{1}{a} e^{ax} + C\right) = e^{ax}$$

$$\int \ln x\, dx = x(\ln x - 1) + C \qquad \frac{d}{dx}\left(x(\ln x - 1) + C\right)$$

$$= \ln x - 1 + x\frac{1}{x} = \ln x$$

Generally we write

$$\int f(x)dx = \underbrace{F(x)}_{\text{a function}} + \underbrace{C}_{\text{constant}}$$

"=" Class of functions

Indefinite integral, cont.

Rules:

$a \neq -1 \quad \int x^a \, dx = \frac{1}{a+1} x^{a+1} + C$

$a = -1, x > 0 \quad \int \frac{1}{x} \, dx = \ln x + C$

$a \neq 0 \quad \int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$

$$\int a f(x) \, dx = a \int f(x) \, dx$$

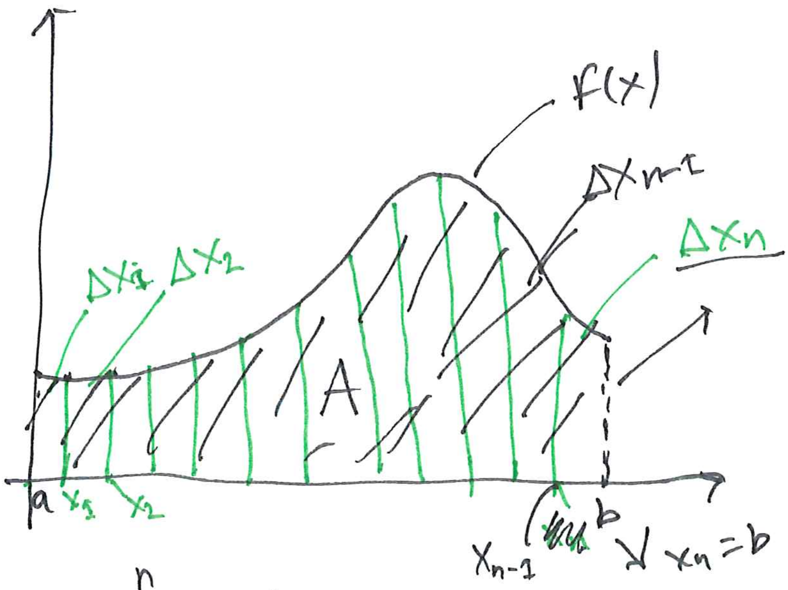
$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx$$

Example: $\int (x^a + b e^{cx}) \, dx = \int x^a \, dx + b \int e^{cx} \, dx = \frac{1}{a+1} x^{a+1} + \frac{b}{c} e^{cx} + B$

Definite integral

We know what $\int f(x) \, dx$ means. What does $\int_a^b f(x) \, dx$ mean?

a, b = limits of integration. It can be interpreted as an area under a curve $A = \int_a^b f(x) \, dx$.



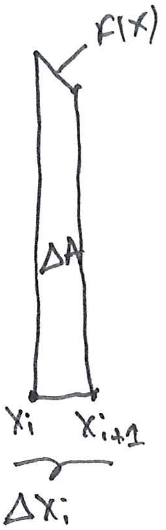
$A(b) =$ "area from a to b"

$\Delta x_n = b - x_n$

$A_n = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$

$x_i^* = x_n$

Suppose $f(x_i) > f(x_{i+1})$



$$f(x_i)\Delta x_i \ge \Delta A = A(x_i+\Delta x_i) - A(x_i) \ge f(x_i+\Delta x_i)\Delta x_i$$

$$\underline{f(x_i)} \ge \underline{\frac{A(x_i+\Delta x_i)-A(x_i)}{\Delta x_i}} \ge \underline{f(x_i+\Delta x_i)}$$

Let $\Delta x_i \to 0$

$$f(x_i) \ge A'(x_i) \ge f(x_i)$$

The derivative of A (area) is the curve's "height", $f(x_i)$.
A must be one of the indefinite integrals of $A'(x_i) = f(x_i)$
we want $A(b)$. Let $F(x) = \int f(x)dx$, we have that

$$A(x) = F(x) + C$$

$$A(a) = 0 = F(a) + C \Leftrightarrow C = -F(a)$$

$$A(x) = F(x) - F(a)$$

$$\underline{A(b) = F(b) - F(a)}$$

$$\boxed{\text{Definite integral: } \int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a)}$$

We can have $a < b$: $\int_b^a f(x)dx = F(a) - F(b) = -\int_a^b f(x)dx$

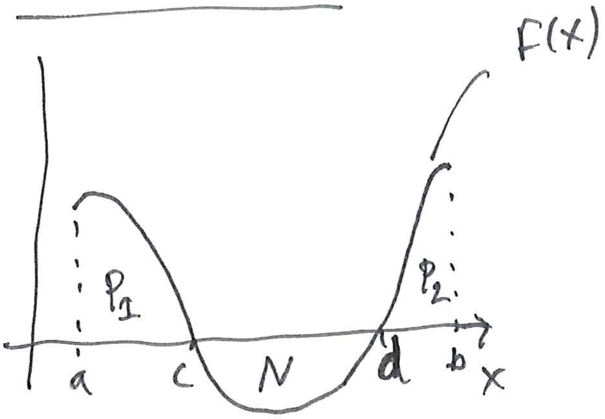
Definite integral is a number, an indefinite integral is a class of functions.

$\int_1^2 (x^2-2)dx$ $\qquad$ $\int (x^2-2)dx = \frac{1}{3}x^3 - 2x + C$

$$\frac{8}{3} - \frac{4\cdot 3}{3} + \frac{2\cdot 3}{3} - \frac{1}{3} = \frac{8}{3} - \frac{6}{3} - \frac{1}{3}$$

$$\int_1^2 (x^2-2)dx = \Big|_1^2 \left(\frac{1}{3}x^3 - 2x + C\right) = \frac{1}{3}\cdot 8 - 4 + \cancel{C} - \left(\frac{1}{3} - 2 + \cancel{C}\right) = \frac{1}{3}$$

When $f(x) < 0$



$$P_1 = \int_a^c f(x)dx > 0$$

$$P_2 = \int_d^b f(x)dx > 0$$

$$N = -\int_c^d f(x)dx > 0$$

We have $A = \int_a^b f(x)dx = P_1 - N + P_2$

Many applications have negative areas, e.g. Cash flows.

Properties of definite integrals

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx \, , \quad \alpha \text{ is a constant}$$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \qquad a < c < b$$

Differentation w.r.t. limits of integration

a = lower limit
b = upper limit

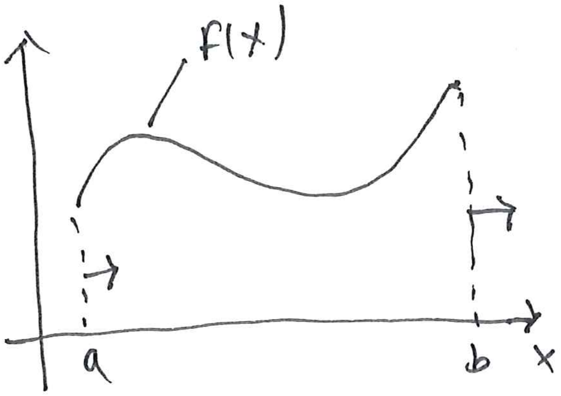
$$\left(\frac{d}{dx}\int f(x)dx = f(x)\right)$$

$$\frac{d}{da}\int_a^b f(x)dx = \frac{d}{da}\Big(F(b) - F(a)\Big) = -f(a)$$

$$\frac{d}{db}\int_a^b f(x)dx = \frac{d}{db}\Big(F(b) - F(a)\Big) = f(b)$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)dx = \frac{d}{dt}\Big(F(b(t)) - F(a(t))\Big)$$

$$= f(b(t))\,b'(t) - f(a(t))\,a'(t)$$



If we increase a: $-f(a)$

If we ~~decrease~~ increase b: $f(b)$

Example: $$\frac{d}{da}\left(\int_a^b (x^2 + 2x)dx\right) = -a^2 - 2a < 0$$

Recap:

Indefinite integrals: $\int f(x)\,dx = F(x) + C \qquad \frac{d}{dx}F(x) = f(x)$

Definite integrals: $\int_a^b f(x)\,dx = F(b) - F(a)$

Value of a bond: $c > 0$, $r > 0$, $p = 0$, T

$$V = \int_0^T e^{-rt} c\,dt = c \Big[-\frac{1}{r} e^{-rt} \Big]_0^T = c\left(-\frac{1}{r}e^{-rT} - \left(-\frac{1}{r}\underbrace{e^{-r\cdot 0}}_{=1}\right)\right)$$

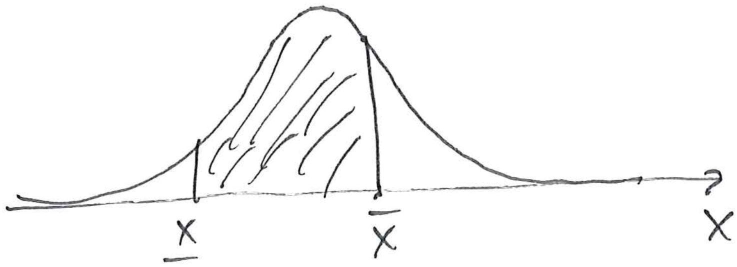
$$= (1 - e^{-rT})\frac{c}{r}$$

$$\frac{d}{dT}\int_0^T e^{-rt} c\,dt = e^{-rT} c \quad (= f(b))$$

iii) Probability:

$$P(\underline{x} \le x \le \bar{x}) = \int_{\underline{x}}^{\bar{x}} f(x)\,dx$$

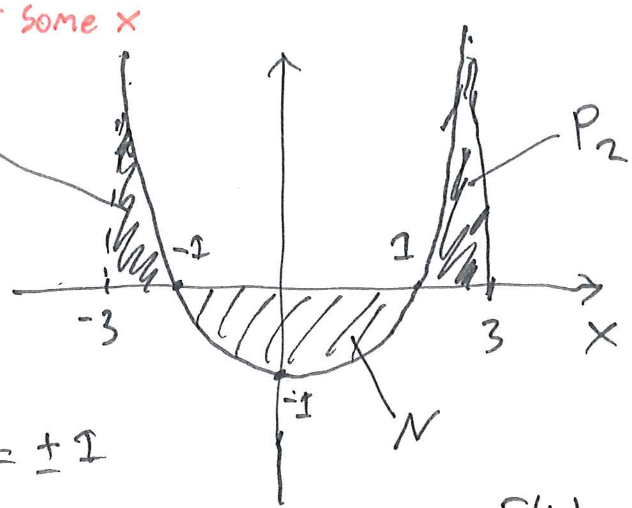
$$= \int_{\underline{x}}^{\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$



= not expressible by elementary functions!

Example: $F(x)$ is negative for some x

$$F(x) = x^2 - 1$$



$$F(x) = 0 \iff x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm\cancel{\sqrt{1}} = \pm 1$$

$F(b) - F(a)$

$$A = \int_{-3}^{3} (x^2 - 1)\,dx = \Big|_{-3}^{3} \left(\tfrac{1}{3}x^3 - x\right)(+C) = \underbrace{\tfrac{1}{3}3^3}_{3^2=9} - 3 - \Big(\underbrace{\tfrac{1}{3}(-3)^3}_{-9} \underbrace{-(-3)}_{+3}\Big)$$

$$= \underbrace{9 - 3}_{6} + \underbrace{9 - 3}_{6} = 12$$

$$P_1 = \int_{-3}^{-1} (x^2 - 1)\,dx = \Big|_{-3}^{-1} \left(\tfrac{1}{3}x^3 - x\right) = \underbrace{\tfrac{1}{3}(-1)^3}_{-\frac{1}{3}} \underbrace{-(-1)}_{+1} - \Big(\underbrace{\tfrac{1}{3}(-3)^3 - (-3)}_{-9+3}\Big)$$

$$= \underbrace{-\tfrac{1}{3} + 1}_{\frac{2}{3}} + \underbrace{9 - 3}_{6} = 6\tfrac{2}{3}$$

$$P_2 = \int_{1}^{3} (x^2 - 1)\,dx = 6\tfrac{2}{3}$$

$$N = -\int_{-1}^{1} (x^2 - 1)\,dx = -\Big|_{-1}^{1} \left(\tfrac{1}{3}x^3 - x\right) = -\Big(\tfrac{1}{3}\cdot 1 - 1 - \big(\overbrace{\tfrac{1}{3}(-1)}^{-\frac{1}{3}} \overbrace{-(-1)}^{+1}\big)\Big)$$

$$= -\Big(\underbrace{\tfrac{1}{3} - 1}_{-\frac{2}{3}} + \underbrace{\tfrac{1}{3} - 1}_{-\frac{2}{3}}\Big) = \frac{4}{3}$$

$$A = P_1 - N + P_2 = 6\tfrac{2}{3} - \tfrac{4}{3} + 6\tfrac{2}{3} = 12\tfrac{4}{3} - \tfrac{4}{3} = 12$$

Integration by parts

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \quad \text{(Product rule)}$$

Take indefinite integrals from both sides:

$$\underbrace{\int \frac{d}{dx}\left(f(x)g(x)\right)dx}_{} = \int f'(x)g(x)\,dx + \int \underline{f(x)g'(x)}\,dx$$

$$= f(x)g(x) \quad (+C)$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$ Integration by parts formula

Does not look very useful, but turns out to be:

- products are difficult to integrate
- we can often rewrite $H(x)$ as $f(x)g'(x)$
- especialy useful if we get $f'(x) = 1$

Example

$$\int \ln x \, dx$$

Choose $f(x) = \ln x \qquad g'(x) = 1$

$f'(x) = \frac{1}{x} \qquad g(x) = x$

$$\int \ln x\,dx = \ln x \cdot x - \int \frac{1}{\cancel{x}}\cancel{x}^{1}\,dx = x\ln x - x + C$$

$$= x(\ln x - 1) + c$$

$$\frac{d}{dx}\left(x(\ln x - 1)\right) = \ln x - 1 + \cancel{x}^{1}\frac{1}{\cancel{x}} = \ln x$$

Integration by parts

$$\int f g' = f g - \int f' g$$

Example: $\int x e^x \, dx$

Choose $f(x) = x \quad g'(x) = e^x$

$f'(x) = 1 \quad g(x) = e^x$

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = x e^x - e^x + C$$

Example:

$$I = \int \ln x \frac{1}{x} dx$$

$f(x) = \ln x \quad g'(x) = \frac{1}{x}$

$f'(x) = \frac{1}{x} \quad g(x) = \ln x$

$$I = \int \ln x \frac{1}{x} dx = (\ln x)(\ln x) - \underbrace{\int \frac{1}{x} \ln x \, dx}_{I}$$

$$I = (\ln x)^2 - I + C_1$$

$$2I = (\ln x)^2 + C_1$$

$$I = \frac{1}{2} (\ln x)^2 + C \qquad C = \frac{1}{2} C_1$$

Integration by parts

$$\int f g' = f g - \int f' g$$

Example:

$\int x^3 e^x \, dx$ Choose $f(x) = x^3$ $g'(x) = e^x$

$f'(x) = 3x^2$ $g(x) = e^x$

$$\int x^3 e^x \, dx = x^3 e^x - \underbrace{\int 3x^2 e^x \, dx}_{?} + C_1$$

$\rightarrow$ $f(x) = 3x^2$ $g'(x) = e^x$

$f'(x) = 6x$ $g(x) = e^x$

$$\int 3x^2 e^x \, dx = 3x^2 e^x - \underbrace{\int 6x e^x \, dx}_{?} + C_2$$

$\rightarrow$ $f(x) = 6x$ $g'(x) = e^x$

$f'(x) = 6$ $g(x) = e^x$

$$\int 6x e^x \, dx = 6x e^x - \int 6 \cdot e^x \, dx + C_3$$

$$= 6x e^x - 6 e^x = 6(x e^x - e^x) + C_3$$

$$\int 3x^2 e^x \, dx = 3x^2 e^x - 6e^x (x - 1) + C_4$$

$$\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6e^x (x-1) + C$$

Integration by substitution

$$\int \underbrace{(x^2+10)}_{u}^{50} \underbrace{2x\,dx}_{du}$$

Solving this directly is extremely cumbersome! Integration by parts fails.

What can we do? Substitute something in place of the original functions.

Let's try: $u = x^2 + 10$ $\qquad \frac{du}{dx} = 2x \Leftrightarrow du = 2x\,dx$

$$\int u^{50}\,du = \frac{1}{51} u^{51} + C = \frac{1}{51}(x^2+10)^{51} + C$$

$$\int \frac{1}{1+x}\,dx \qquad \int \frac{1}{x}\,dx = \ln x + C$$

$u = 1+x \qquad \frac{du}{dx} = 1 \qquad du = dx$

$$\int \frac{1}{u}\,du = \ln u + C = \underline{\ln(1+x) + C}$$

Integration by substitution

How to find $\int G(x)dx$?

1. Pice out a "part" of $G(x)$ as a new variable $u=g(x)$
2. $du = g'(x)dx$ and find $dx = \frac{du}{g'(x)}$
3. Substitute $u=g(x)$ and $dx=\frac{du}{g'(x)}$ to get from $\int G(x)dx$ to $\int f(u)du$
4. Find $\int f(u)du = F(u)+C$
5. Replace u by $g(x)$ to get $F(g(x))+C$

Example: $\int x^3\sqrt{1+x^2}\,dx$

1. $u=\sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}}$
2. $du = \cancel{\tfrac{1}{2}}\underbrace{(1+x^2)^{-\frac{1}{2}}}_{\frac{1}{u}}\cancel{2}x\,dx \iff u\,du = x\,dx$
3. $(u^2-1)u = (1+x^2-1)\sqrt{1+x^2} = x^2\sqrt{1+x^2}$

$$\int \underbrace{x^2\sqrt{1+x^2}}_{(u^2-1)u}\cdot\underbrace{x\,dx}_{u\,du} = \int (u^2-1)u^2du = \int(u^4-u^2)du$$

4. $= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$
5. $\int x^3\sqrt{1+x^2}\,dx = \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$

Substituting more than once

Example: $\displaystyle\int \frac{1}{x\ \ln(x)\ \ln(\ln(x))}\,dx$, $\quad x>0$

$v = \ln(x) \qquad dv = \frac{1}{x}dx$

$$\int \frac{1}{\ln(x)\ \ln(\ln(x))}\frac{1}{x}\,dx = \int \frac{1}{v}\frac{1}{\ln(v)}\,dv$$

$u = \ln(v) \qquad du = \frac{1}{v}dv$

$$\int \frac{1}{\ln(v)}\frac{1}{v}\,dv = \int \frac{1}{u}\,du = \ln(u) + C$$
$$= \ln(\ln(v)) + C$$
$$= \ln(\ln(\ln(x))) + C$$

Integration by substitution: definite integrals

You need to change the limits of integration as well as the integral!

Example: $\int_{2}^{3} e^{x^2} \underbrace{2x\,dx}_{du}$

$u = x^2 \quad du = 2x\,d\cancel{x}$
$2x\,dx$

$u(3) = 3^2 = 9$
$u(2) = 2^2 = 4$

$\Rightarrow \int_{4}^{9} e^{u}\,du$

Two options:

i) Calculate the indefinite integral first and the substitute in $u = g(x)$.

(ii) Do the full substitution including limits!)