Mathematics 2 #1.

<u>Functions</u>. What is a function?

Concept: Given two sets D and E, a function from D to E is a rule that ~~to~~ to each element in D, assigns <u>one</u> (precisely!) element in E.

Ex If {your position, time til exam starts, weather}
uniquely determines your means of transportation
$\in$ {on foot, bike, bus, tram, metro, taxi, stay home & skip it}
that defines a function.

A function <u>need not</u> be described as
$f(x) =$ [something with x].

Question for you: Will the following define a function

You utilize (K, L) to produce $F(K, L)$ units of one final good, sold at unit price 1.

Unit costs c, w. You choose (K, L) to maximize $F(K, L) - cK - wL$.

Q: (K, L) function of (c, w) ?

A: ....

In Math 2, functions will output <u>numbers</u>

(note: ω is not a number).

So in Math 2, we will say that: provided K and L unique for each (c, w):

Two functions, $K = K(c, w)$, $L = L(c, w)$.

A function in Math 2 will take as <u>input</u> an n-tuple $(x_1, \dots, x_n)$ of numbers.

(Focus on $n = 2$ (and $n = 1$).)

Sometimes denoted

$\mathbf{x}$ (bold, book)

$\vec{x}$ (notes)

Continuous functions (nearly all in Math 2).

Fix a point $\vec{a}$ in the domain D of f.

f is continuous at $\vec{a}$ if:

Given any (however small) positive error margin ε around the true value $f(\vec{a})$, I can meet that by ~~at~~ merely restricting $\vec{x}$ to be close to $\vec{a}$.

"Example": if f is continuous at $1/3$, then you can get arbitrarily close by $f(\underbrace{0,3333\ldots3}_{d \text{ digits}})$.

"Non-example": $\operatorname{sgn} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

Some functions of a single variable x

- polynomials $c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$
- power functions x^r, r any constant
- exponential functions a^x, $a > 0$ constant
- logarithms: $\log_b x$

All continuous wherever defined.

- Compositions of continuous functions, ex: $2^{x^e + 3x}$, $x > 0$

continuous wherever defined.

The intermediate value theorem

"on" = on the entire.

Let f be continuous [of a single variable] on interval $[a, b]$.

Then f attains all values between $f(a)$ and $f(b)$.

Ex: Does $x^5 + x + x^2$ have a zero?

Yes. $f(-1) = -1 < 0$. $f(1) = 3 > 0$.

Does not say where. Does not tell whether one or many.

. Claim w/o proof?

OK to claim continuity as long as correct.

Inverse functions: If to each y $= f(x)$ value there is $\underbrace{\text{only one } x \text{ value}}$, the function f is one-to-one ("horizontal line test") and has an inverse f^{-1}.

Derivative? Have

$$f(f^{-1}(x)) = x$$

so

$$f'(f^{-1}(x)) \cdot (f^{-1})' = 1$$

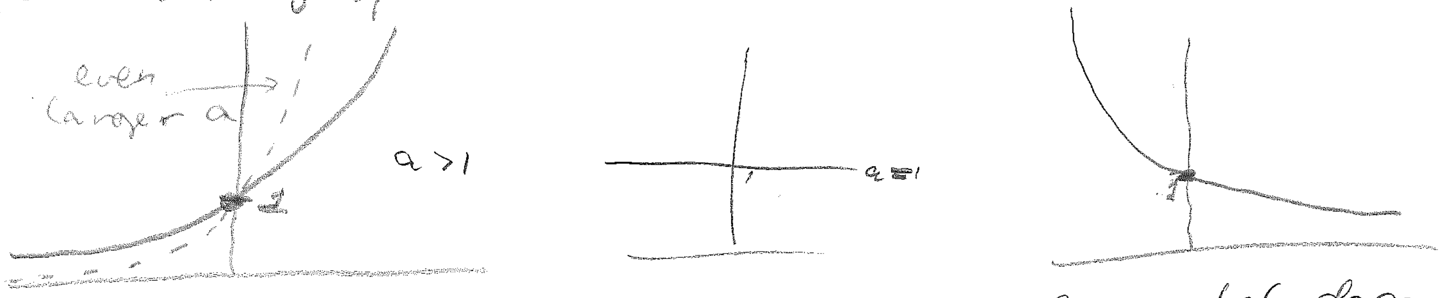
$$(f^{-1})'(x) = 1 / f'(f^{-1}(x))$$

Exponential functions: Constant $\cdot$ a^x

put this $= 1$.

$a > 0$, base number.

$f(x) = a^x$, graph:



exponential growth

exponential decay

For $a > 1$: strictly increasing (more in $\Rightarrow$ more out) and has an inverse: $\log_a$

For $a < 1$: strictly decreasing (more in $\Rightarrow$ less out)

. . . . ($\log_a$.)

For $a \neq 1$: a^x attains all positive values.

Q: $\frac{d}{dx} a^x$? $\frac{d}{dx} \log_a x$?

TL;DR: $a^x \ln a$ resp $\frac{1}{x \ln a}$ where $\ln = \log_e$,

$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$. $\approx 2.718281828459045$

$$\frac{d}{dx} a^x = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x f'(0).$$

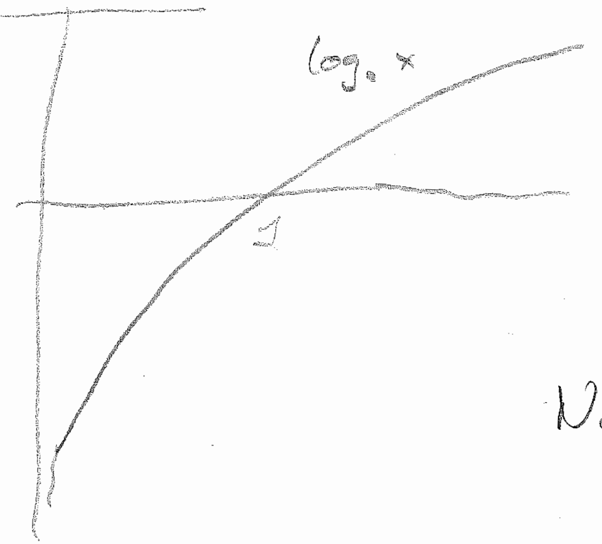
The number e: <u>the</u> base number that makes $f'(0) = 1$.

Now, because of function / inverse function:

$$a^{\log_a x} = x \qquad e^{\ln a} = a \; \ldots\ldots$$

Can write $a^x = e^{x \ln a}$. with derivative $\underbrace{e^{x\ln a}}_{a^x} \ln a$

<u>The logarithm</u>: Let $a > 1$. Graph:



Fact: $\boxed{\log_a x = \frac{\ln x}{\ln a}}$

So for $\frac{d}{dx} \log_a x$, we only need $\frac{d}{dx} \ln x$.

Now $e^{\ln x} = x$. $\frac{d}{dx}$ both sides:

$$\underbrace{e^{\ln x}}_{x} \cdot (\ln x)' = 1$$

so $\boxed{(\ln x)' = \frac{1}{x}}$

Some rules for logs:

$$\log_a (x y^z) = \log_a x + \log_a y^z = \log_a x + z \log_a y.$$

$$\log_a (x/y) = \log_a x - \log_a y \qquad (\text{put } z = -1)$$

$$\log_a (x+y) = \text{no simple rule! Do not try}$$

to write as sum of logs...

$$\log_a 1 = 0 \quad , \quad \log_a a = 1 .$$

Logarithmic differentiation: If $f(x) > 0$,

then $\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} f'(x)$, so $f'(x) = f(x) (\ln f(x))'$

$$\text{Ex: } \frac{d}{dx} x^{-x} = x^{-x} \cdot \frac{d}{dx}(-x \ln x) = \ldots$$

$$\frac{\partial}{\partial K} K^{\alpha} L^{\beta} = K^{\alpha} L^{\beta} \cdot \frac{\partial}{\partial K}(\alpha \ln K + \beta \ln L) = \frac{\alpha}{K} K^{\alpha} L^{\beta}$$

In the interest of time:
Rules for exponentials (a^{p+q} etc.) only mentioned, not listed.

(Here:
End of lecture 1, start lecture 2.)

Compound interest

Q: A credit card charges 2% per month.

What is the effective annual rate?

We have : $K(1) = K_0 \, 1.02^{12}$. Note, $1.02^{12} > 1.24$

year 1

A: $(1.02^{12} - 1) \cdot 100\%$

Q: 'A credit card charges "25% effective annual rate",

but compounds interest monthly.

What is the monthly rate p in %?

$$\cancel{K_0} \left(1 + \frac{p}{100}\right)^{12} = 1.25 \cancel{K_0}$$ Take $\sqrt[12]{}$

$$1 + p/100 = (1.25)^{1/12}$$ and ....

—

Compare $\left(1 + \frac{r}{n}\right)^n$: large n vs small n, $r > 0$ fixed

larger!

Increasing in n.

What happens as $n \to +\infty$? Fact: $\to e^r$

"Continuously compounded interest".

Q: What does "at rate 0.24/year" mean?

1.24^t ? or $e^{0.24t}$?

A_1: Read the context.

A_2: Don't worry in Math 2.

Max and min

Fix a function f.

A maximum (resp. minimum) for f is a point x^*

such that $f(x^*) \geq f(x)$, all x

resp $\leq$

It is strict if ">" resp "<", all $x \neq x^*$

Note, terminology: max/min is the input.

The maximal/minimal f is the maximum value

resp minimum value.

We also use notation like

$$V(r) = \max_x f(x, r)$$

"value function"

exogenous variable

Analogously for $\vec{x} = (x_1, \ldots, x_n)$, $\vec{x}^* = (x_1^*, \ldots, x_n^*)$

Local max/min (strict max/min):

x^* is a local max/min if there exists a neighbourhood U of x^* – i.e. a small open set centered at x^* – such that when we restrict f to U, x^* becomes a max resp min.

no boundary points included

Addendum: analogous for several variables. Just put boldface or overarrow on x.

Ex:

To distinguish: "previous page" max/min $\to$ "global max/min"

Note, a global max/min is also local.

Terminology: (local) extreme point: (loc.) max or min

extreme value: $f($ $\underbrace{\quad\quad\quad\quad\quad\quad}$ $)$

(local)

<u>Questions about max / min.</u> Fix a function f.

→ Do we know whether any max/min exist(s)?

→ What could they be like? Behaviour "on the graph".

↳ ... in Math 2?

→ How to find a short list of possible points?

→ And then, how to decide which of these are indeed max / min points?

→ If. $V(r) = \max_{\vec{x}} f(\vec{x}, r)$: Behaviour of V? $V'(r)$?

For existence, the <u>extreme value theorem</u>:

Let $S \subseteq \mathbb{R}^n$ be nonempty, closed and bounded.

closed: contains all of its boundary points

bounded: can fit in a bounded box in $\mathbb{R}^n$

If f is continuous on S, it has both a max over S and a min over S

Non-examples:

$\ln x$ on $[0, 1]$

$K^\alpha L^\beta$ on $\{(K, L);\ K \geq 0,\ L \geq 0\}$

Example: Let $\overset{c > 0}{\alpha \geq 0}$, $\overset{w > 0,\ m > 0}{\beta \geq 0}$. $K^\alpha L^\beta$ on the set

$\{(K, L);\ K \geq 0,\ L \geq 0,\ cK + wL = m\}$

or on the set

$\{ \qquad \leq m\}$

So: The problems

. max $K^\alpha L^\beta$ s.t. $cK + wL = m$

and

. . . . $\leq m$

both have solution.

* What kind of points?
  - Discontinuity points
  - Non-differentiability points
  - Stationary points: $\frac{\partial f}{\partial x_i} = 0$, all i
  - Boundary points

    ↓ note: endpoints for single-variable optimization.

Math 2 relevance?

— ... no.

$|x|$ has min for $x=0$.

YES!

YES. Lagrange / Kuhn-Tucker.

* How to find: $\underbrace{\text{Stationary points, Lagrange points on boundary}}_{\text{Kuhn-Tucker cond's}}$

* How to tell? 2^{nd} derivatives / 2^{nd} order cond's

  Concavity / convexity

  Possibly: If we have existence: find all & compare.

* $V(r)$? Envelope theorem ("omhyllingssetningen").

Single variable: $f(x)$, f continuously differentiable defined on an interval.

Candidate points for max/min:

- endpoints if f defined there
- stationary points: where $f'(x) = 0$.

Example: $f(x) = e^{-x^2}$ on the set $[-1, 2]$. ← [ ] means: endpoints included

Candidate points for max/min:

$x = -1$

$x = 2$

any x s.t. $-2x e^{-x^2} = 0$, i.e. $x = 0$.

Lect. 3) Then the "how to tell" part. Various methods

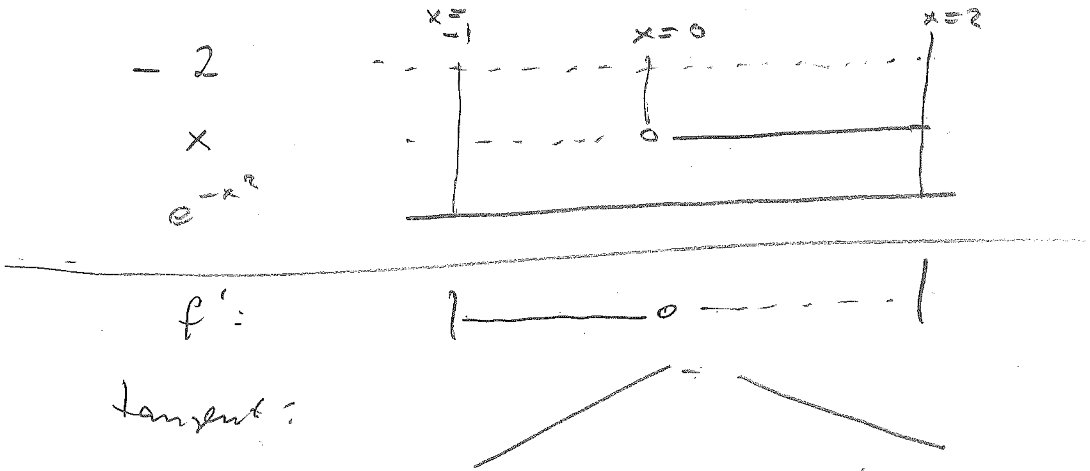
- The extreme value theorem applies here. Evaluate $f(-1)$, $f(0)$ (largest), $f(2)$ (smallest)
  $x=0$ glob. max $x=2$ glob. min
- The first-derivative test   Note: "single variable only"
- The second-derivative test.

The first-derivative test (single variable only)

Check the sign of f'.

Write f' as factors/ratios, sign diagram.

The example: $f'(x) = -2x \underbrace{e^{-x^2}}_{>0}$.



This diagram tells us: strict glob. max for $x=0$.
loc. min for $x=-1$, $x=2$

Diagram does not tell which point is glob. min.

Evaluate $f(2) = e^{-4} < f(-1) = e^{-1}$.

Then $x=2$ is strict global min, since:

For $x \in [0,2)$: $\searrow$, so $f(x) > f(2)$

For $x \in [-1,0]$: $\nearrow$ so $f(x) \geq f(-1) > f(2)$

Second derivatives (>1 variable will be more involved)

f'' = the derivative of the derivative

- If $f'' \leq 0$ everywhere, f is <u>concave</u> and any stationary point will be global maximum.

- If $f'' \geq 0$ ...... <u>convex</u> and .... glob. min

Local version: Let $f'(x^*) = 0$

If $f''(x^*) < 0$, resp > 0

then x^* is strict local max

resp strict local min.

From the example $f(x) = e^{-x^2}$, $f'(x) = -2x\,e^{-x^2}$

stationary pt: $x = 0$. $f''(0) = -2e^{-0^2} = -2 < 0$.

$\Rightarrow$ strict loc. max.

(The question about value function properties: later.)

<u>n variables</u> (Main focus: $n = 2$.)

- Existence: Extreme value theorem
- What points & how to find:
  - → boundary: ⇒ Lagrange / Kuhn-Tucker, later.
  - → interior points: stationary. $\frac{\partial f}{\partial x_i}(\vec{x}^*) = 0$, all i.
- Classify: concavity / convexity; second derivatives.

Classification:

Fact: If f is concave (resp. convex) then any stationary point is glob. max (resp. glob. min)

Shall cover:

- 2^{nd} derivatives criteria for $n=2$
- When $n \geq 2$: sum of concave (resp convex)

Criteria for concavity / convexity, $n = 2$.

Let $h(x,y) = f''_{xx}(x,y)\, f''_{yy}(x,y) - (f''_{xy}(x,y))^2$

If, everywhere:

$$\left.\begin{array}{l} h \geq 0 \\ f''_{xx} \geq 0 \\ f''_{yy} \geq 0 \end{array}\right\} \Rightarrow f \text{ convex} \qquad \left.\begin{array}{l} h \geq 0 \\ f''_{xx} \leq 0 \\ f''_{yy} \leq 0 \end{array}\right\} \Rightarrow f \text{ concave}$$

Ex: $f(x,y) = x^4$ is convex. (Note: $f''_{yy} \equiv 0 \equiv f''_{xy}$)

Local 2^{nd} order cond's: Let (x^*, y^*) be a stationary point.

- If $h(x^*, y^*) > 0$ then (x^*, y^*) is a strict local extreme point:
  - a strict loc. max if $f''_{xx}(x^*, y^*) < 0$
  - a strict loc. min if $f''_{xx}(x^*, y^*) > 0$
- If $h(x^*, y^*) < 0$ then (x^*, y^*) is a <u>saddle point</u>:
  Saddle point = stationary point that is neither loc. max nor loc min.
- "no conclusion" if $h(x^*, y^*) = 0$.

Classifying stationary points – ambition for the (2019) exam (start lecture no. 4)

To do away with the most nitpicking details, this clarifies what you would be expected to manage on an exam. Suppose we have found a stationary point $\mathbf{x}^*$ to classify, for a C^2 function f of n variables.

(I) If f is concave (resp. convex) – and can with «reasonable» skill and effort (see below!) be shown to be so – then you are expected to conclude global max (resp. global min) even if the (II) below fails to conclude. Tools[1]:

- If $n = 1$: Second derivative. $f''(x) \leq 0$ everywhere $\iff f$ concave. $f''(x) \geq 0$ everywhere $\iff f$ convex.
- If $n = 2$: Let $a = f''_{xx}$ and $c = f''_{yy}$ and $h = f''_{xx}f''_{yy} - \left(f''_{xy}\right)^2$. Convex $\iff a \geq 0$, $c \geq 0$ and $h \geq 0$ all hold everywhere. Concave $\iff a \leq 0$, $c \leq 0$ and $h \geq 0$ all hold everywhere. Note «$h \geq 0$» for both.
- Any n: sums of concaves are concave, and sums of convexes are convex.

(II) If item (I) does not apply, then we have the following tools:

- If $n = 1$: first derivative test, and/or $f''(x^*)$ (evaluate at the point).
- If $n = 2$: Let $A = f''_{xx}(x^*, y^*)$, $B = f''_{xy}(x^*, y^*)$ and $C = f''_{yy}(x^*, y^*)$ – note, we have inserted for the stationary point and gotten three numbers A, B and C.
  - If $AC - B^2 > 0$, then either strict local max (if $A < 0$) or strict local min (if $A > 0$).
  - If $AC - B^2 < 0$, then saddle point (i.e. neither loc. max. nor loc. min)
  - if $AC = B^2$: you can stop and declare «no conclusion».

Note this fact: If f has more than one stationary point, but finitely many – then it cannot be concave nor convex, and you can go to (II)! (Indeed, if a convex/concave function has two stationary points, then all the points in between are stationary.)

What is that «reasonable» thing? Two examples:

Example 1: $f(x, y) = (x-y)^4$ has global minimum along the line $y = x$ (f is zero there and positive otherwise), but let us use the second derivatives to illustrate: $f''_{xx}(x, y) = f''_{yy}(x, y) = 12(x - y)^2$ and so both f''_{xx} and f''_{yy} are everywhere nonnegative. Also, $f''_{xy}(x, y) = -f''_{xx}(x, y)$, so we have $f''_{xx}f''_{yy} - \left(f''_{xy}\right)^2 = 0$ everywhere. Therefore f is convex, and the entire line of stationary points are global minima. Note, «$AC - B^2$» is zero, but we can still conclude!

Example 2: $F(x, y, z) = f(x, y) + g(z)$, f as in Example 1, $g(z) = e^{z\cdot(z-2)}$. If g is convex and has a stationary point z^*, we have global minimum at (t, t, z^*) for every $t \in \mathbf{R}$ (that is, on a line from $(0, 0, z^*)$ in the $y = x$ direction, $z = z^*$ fixed). Now, $g'(z) = 2(z - 1)e^{z\cdot(z-2)}$ (stationary point for $z = 1$) and $g''(z) = 2(2z^2 - 4z + 3)e^{z\cdot(z-2)}$. Since $2z^2 - 4z + 3 > 0$, g is convex.

[1] Require f to be defined on a so-called *convex set*, but let's just assume no holes in the domain of f. For $n = 1$: interval!

Example: Find and classify the stationary points of $f(x,y) = (x^2-1)^2 + (x^2y-x-1)^2$.

$f'_x(x,y) = 2(x^2-1)\cdot 2x + 2(x^2y-x-1)\cdot(2xy-1)$

$f'_y(x,y) = 2(x^2y-x-1)\cdot x^2$.

$f'_y(x,y)=0 \iff x^2=0$ or $x^2y-x-1=0$.

Case $x=0$. Then $f'_x(0,y) = 2\cdot(-1)\cdot(-1) \neq 0$
no st.pt here!

Case $x^2y-x-1=0$. then $f'_x(x,y) = 4x(x^2-1)$.
Three subcases: $x=0$ (impossible!), $x=1$, $x=-1$.
$x=1$: Then $y=2$; $x=-1$: then $y=0$.

Two points: $(-1,0)$ and $(1,2)$.

To classify: Wolfram Alpha with the query

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Hessian matrix of (function expression)
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returns the second derivatives in a matrix, and the "Hessian determinant" is the AC-B^2 thing. Click here for example.

$$f''_{xx}(x,y) = 4y\cdot\underbrace{(x^2y-x-1)}_{=0} + 4\underbrace{(x^2-1)}_{=0} + 8\underbrace{x^2}_{=1} + 2(2xy-1)^2$$

At stationary pts:

so we get $8 + 2(2xy-1)^2 \geq 8 > 0$

($\Rightarrow$ not loc max!)

$$f''_{xy}(x,y) = 2x^2\underbrace{(2xy-1)}_{\text{This I forgot on the board!}} + 4x\underbrace{(x^2y-x-1)}_{=0 \text{ at st. pt}}$$

CORRECTION

$f''_{yy}(x,y) = 2x^4$ which $\underset{>0 \text{ too}}{=2}$ at the stationary points

So the $AC - B^2 = 16 + 4(2xy-1) - (2(2xy-1))^2 = 16 > 0.$

Both are <u>strict local min</u>.

—

Indeed, they are global min: $f(x,y) = \overset{\geq 0}{(x^2-1)^2} + \overset{\geq 0}{(x^2y-x-1)^2}$

with global min $\Leftrightarrow$ $x^2-1=0$ AND $x^2y-x-1=0$

– precisely those two points!

—

Note: Two strict local min & no other stationary points

– counterintuitive? Would be impossible if $n=1$ variable!

Plot: click here for link to Wolfram Alpha

Properties of the value function

Ex: We solved $\max\left(e^{-x^2} - rx\right)$ for $r = 0$.

For $r \approx 0$: $V(r) \approx \underset{=1}{V(0)} + r\,V'(0)$.

Can I find $V'(0)$ without solving $V(r)$ for general r?

Yes.

The envelope theorem: Suppose $\vec{x}^* = \vec{x}^*(r)$ solves
$$\max_{\vec{x}}/\min_{\vec{x}} f(\vec{x}, r)$$
every fixed r.

$$V(r) = f(\vec{x}^*(r), r). \quad \text{Then}$$

$$V'(r) = \underbrace{\cancel{f'_1} \cdot \frac{dx_1^*}{dr} + \dots + \cancel{f'_n} \cdot \frac{dx_n^*}{dr}}_{=0 \text{ by the FOC.}} + \frac{\partial f}{\partial r}(\vec{x}^*, r).$$

Also true for saddle points.

For single variable: Also true for endpoint max/min.

In the example: $V'(r) = \frac{\partial}{\partial r}\left(e^{-x^2} - rx\right)\Big|_{x=x^*} = -x^*.$

$$V'(0) = x^*(0) = \underline{\underline{0}}$$

Constrained max/min. Today: equality constraints.

Problem: $\max_{\min} f(\vec{x})$ subject to $\begin{cases} g_1(\vec{x}) = b_1 \\ \vdots \\ g_m(\vec{x}) = b_m \end{cases}$

$\vec{x} = (x_1, \ldots, x_n)$, $n > m$

Lagrange's method produces candidate points for all? Not really, but in Math 2: Take it as necessary F.O.C. max/min.

How to: · Form the Lagrangian

$$L(\vec{x}) = f(\vec{x}) - \lambda_1(g_1(\vec{x}) - b_1) - \ldots - \lambda_m(g_m(\vec{x}) - b_m).$$

Conditions: $\frac{\partial L}{\partial x_i} = 0$, all $i = 1 \ldots n$

$g_j(\vec{x}) = b_j$ all $j = 1, \ldots, m$

(Tip: eliminate the λ_i.)

Example:

$$\max/\min \quad x^2+y^2+z^2 \quad \text{subject to} \quad \begin{aligned} x^2+y^2+2z^2 &= 4 \\ x+y+z &= 3 \end{aligned}$$

- Show that these problems have a solution, and find them using Lagrange's method.

Continuous function, closed and bounded set $\neq \emptyset$ ← $C(1,1,1)$ is admissible

if $|x|$ or $|y|$ or $|z|$ too large, then not admissible

$\Rightarrow$ max & min both exist

To find: $L(x,y,z) = x^2+y^2+z^2 - \lambda(x^2+y^2+2z^2-4) - \mu(x+y+z-3)$

Conditions:

$$0 = 2x - 2\lambda x - \mu \quad (I)$$
$$0 = 2y - 2\lambda y - \mu \quad (II)$$
$$0 = 2z - 4\lambda z - \mu \quad (III)$$
$$x^2+y^2+2z^2 = 4 \quad (IV)$$
$$x+y+z = 3 \quad (V)$$

subtract I from II and III

(II') $0 = 2(y-x) - 2\lambda(y-x)$ which $= 2(1-\lambda)(y-x)$

(III') $0 = 2(z-x) - 2\lambda(2z-x)$

(II') $\Leftrightarrow$ $y = x$ or $\lambda = 1$.

- Case $y = x$. (IV) then becomes $2x^2 + 2z^2 = 4$
  (V) — " — $2x + z = 3$

  Insert $z = 3 - 2x$ into $x^2 + z^2 = 2$

  $x^2 + 9 - 12x + 4x^2 = 2$

  here was the error!

CORRECTION
Corrected!

$5x^2 - 12x + 7 = 0 \Leftrightarrow x = \frac{1}{10}(12 \pm \sqrt{144-140})$

$x_1 = 1$ yields $y_1 = 1$ and $z_1 = 3 - 2x_1 = 1$. $f(1,1,1) = 3$.

$x_2 = \frac{7}{5}$ yields $y_2 = \frac{7}{5}$ and $z_2 = \frac{15-14}{5} = \frac{1}{5}$. $f(\frac{7}{5}, \frac{7}{5}, \frac{1}{5}) = \frac{103}{25}$ $(>4>3)$

CORRECTION
Corrected here too!

- Case $\lambda = 1$; (III') yields $2(z-x) - 2(2z-x) = 0 \Leftrightarrow \underline{z=0}$.

  With $z=0$, constraints $x^2 + y^2 = 4$ & $x + y = 3$.

  Insert: $x^2 + 9 - 6x + x^2 = 4$, "$b^2 - 4ac$" $= 36 - 40$

  No solution here!

So: $\underline{\text{min for } (x,y,z) = (1,1,1)}$

$\underline{\text{max for } (x,y,z) = \left(\frac{7}{5}, \frac{7}{5}, \frac{7}{5}\right)}$