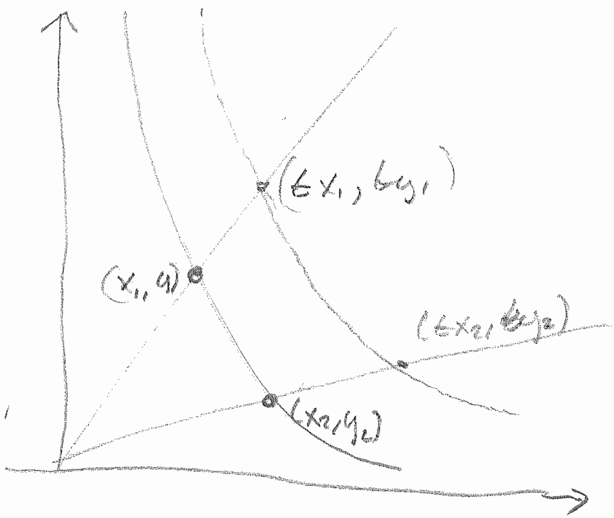
Recall that for homogeneous functions:



"Know one level curve, know them all":

$\circledast$ If $\vec{u}$ and $\vec{v}$ are on the same level curve $f = C$
then, each $t > 0$:
$t\vec{u}$ and $t\vec{v}$ are on the same level curve $f = D$.

There are functions that have this property $(*)$ but that are not homogeneous.

Def: A function f is homothetic if $\circledast$ holds.

[The domain D must be so that if $\vec{x} \in D$ then $t\vec{x} \in D$, all $t > 0$]

Alternatively: if f is a one-to-one [i.e. invertible] transformation of a homogeneous.

f homothetic

In "most" applications – and for Math 2 exam purposes – we only consider homothetic f that are
strictly increasing transformations of homogeneous functions

Example:

$$f(x_1, \dots, x_n) = a_1 \ln x_1 + \dots + a_n \ln x_n .$$

Why? $e^{f(\vec{x})} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ (Cobb – Douglas)

is homogeneous, degree $a_1 + \dots + a_n$.

or from the definition:

$$f(t\vec{x}) = a_1 \ln(tx_1) + \dots + a_n \ln(tx_n)$$
$$= a_1 \ln x_1 + \dots + a_n \ln x_n + (a_1 + \dots + a_n) \ln t .$$

If $f(\vec{u}) = f(\vec{v})$ then

$$f(t\vec{u}) = f(\vec{u}) + (a_1 + \dots + a_n) \ln t$$
$$= f(\vec{v}) + \text{same} = f(t\vec{v}).$$

Example $\left(\ln\left(1 + \sqrt[35]{x_1^2 + x_2^2 + \dots + x_n^2}\right)\right)^3$

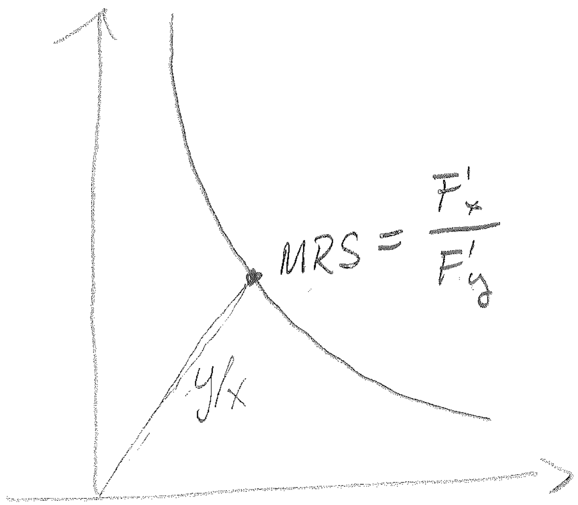
- Recall differentials: $d\ln x = \frac{1}{x} dx$ etc.

- Example: For $f > 0$, $x > 0$: $El_x f = \dfrac{d\ln f}{d\ln x}$

The elasticity of substitution

Consider a level curve $F(x,y) = C$ for F

assume: decreasing in the xy plane and convex



Question:

More along the level curve so much that the MRS changes by 1%.

How much does relative factor use y/x change?

Def.: The elasticity of substitution between y and x

is $\sigma_{y,x} = El_{MRS}\ y/x$ (along the level curve)

$$= \frac{d \ln y/x}{d \ln \frac{F'_x}{F'_y}}$$

Fact: $\sigma_{y,x} = \sigma_{x,y}$ because $\frac{d \ln x/y}{d \ln F'_y/F'_x} = \frac{d(-\ln y/x)}{d(-\ln F'_x/F'_y)}$

How to calculate?

- Formula, or
- Manipulate differentials

Next example: the latter. (Exercise: same function, use the formula)

Example: The CES class of functions,

$$F(K,L) = A \cdot (aK^{-Q} + bL^{-Q})^{-m/Q}$$

Show that σ_{LK} is constant.

Proof: $$F'_K = A \cdot \left(-\frac{m}{Q}\right)(aK^{-Q} + bL^{-Q})^{-\frac{m}{Q}-1} \cdot a \cdot (-Q) K^{-1-Q}$$

$$F'_L = A \cdot \left(-\frac{m}{Q}\right)(aK^{-Q} + bL^{-Q}) \qquad b \cdot (-Q) L^{-1-Q}$$

$$MRS = \frac{a}{b} \cdot \left(\frac{L}{K}\right)^{Q+1}$$

$$\sigma = \frac{d \ln L/K}{d \ln MRS} = \frac{d \ln L/K}{d\left(\ln \frac{a}{b} + (Q+1) \ln \frac{L}{K}\right)} = \cancel{\frac{d \ln L/K}{(Q+1)\, d \ln L/K}}$$

$$= \frac{1}{Q+1}$$

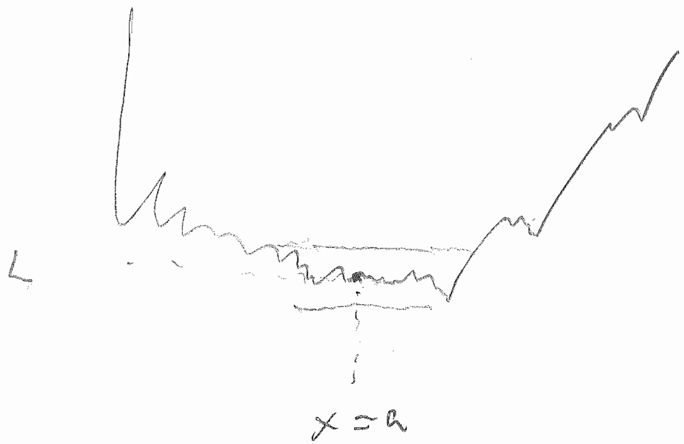
Limits

Q1: What do we mean by $\lim\limits_{x \to a} f(x) = L$

[or: $\lim\limits_{x \to a^+}$, $\lim\limits_{x \to -\infty}$, ...]

Q2: How to compute?

A1: We can get (the value of) $f(x)$ as close to L as we might want to, by merely restricting x to be close to a.



Interpretation:
"error margins"

- One-sided limits: $\lim_{x \to a^+}$ from the right

  $\lim_{x \to a^-}$ from the left.

- As $x \to +\infty$: "large and positive" in place of "close to a"

  As $x \to -\infty$: "large and negative"

Terminology: We say that $\lim_{x \to a} f(x)$ ($\text{or} \to -\infty$, or ...) exists

if $L = \lim_{x \to a} f(x)$ is a well-defined number.

Also we say: $f(x)$ converges to L as $x \to a$

If it doesn't converge to any $L \in \mathbb{R}$: diverges

Still we write, e.g. $\lim_{x \to 0} x^{-2} = +\infty$

[Sometimes you can find "exists or $= +\infty$"...]

Note: $\lim_{x \to 0} \frac{1}{x}$ does not "diverge to $\pm\infty$"

" " " "diverge to $-\infty$"

(But it surely diverges!)

A2: How to compute?

- Continuous functions (at $a \in \mathbb{R}$)

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)$$

Ex: $\lim_{x \to 4} \frac{x-1}{x^2+1} = \frac{4-1}{4^2+1} = \underline{\underline{\frac{3}{17}}}$

- As long as both $L = \lim_{x \to a} f(x)$ and $M = \lim_{x \to a} g(x)$ exist:

same a

sums / differences: $\lim_{x \to a} \left(f(x) \pm g(x)\right) = L \pm M$

products: $\lim_{x \to a} \left(f(x)\, g(x)\right) = L\, M$

ratios? "OK" if L, M also $\neq 0$; more later!

- If we know that $L = \lim_{x \to a} f(x)$ exists but don't know about $\lim_{x \to a} g(x)$

Sums/differences: "still valid formula":

$$\lim_{x \to a} \left(f(x) \pm g(x) \right) = L \pm \lim_{x \to a} g(x)$$

- If $\lim_{x \to a} g(x)$ exists: OK.
- If $\lim_{x \to a} g(x) = +\infty$: $L \pm \infty$
  $-\infty$: $L \mp \infty$
- If diverges otherwise: so does $\lim (f \pm g)$

products: still valid provided $L \neq 0$.

Beware "$0 \cdot \pm\infty$".

ratios: Beware "$\frac{0}{0}$" and "$\pm \frac{\infty}{\infty}$".

"Indeterminate forms": limits that $\longrightarrow$ "$\frac{0}{0}$"

or "$\frac{\infty}{\infty}$" or "$\infty \cdot 0$" or "$\infty - \infty$" or...

Ex: What is $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^3 - 5x^2 + 3x + 9}$?

Try to insert $x = 3$: yields $\frac{0}{0}$.

For polynomials: If $p(a) = 0$ then

$p(x) : (x-a)$ is polynomial.

Long division: written highest-order to lowest

$$(x^2 - 9) : (x - 3) = \underline{x + 3}$$

$$\begin{array}{r}
\underline{x^2 - 3x} \\
0x^2 + 3x - 9 \\
\underline{3x - 9} \\
0
\end{array}$$

$$(x^3 - 5x^2 + 3x + 9) : (x - 3) = \underline{x^2 - 2x - 3}$$

$$\begin{array}{r}
\underline{x^3 - 3x^2} \\
-2x^2 + 3x + 9 \\
\underline{-2x^2 + 6x} \\
-3x + 9
\end{array}$$

So $$\frac{x^2-9}{x^3-5x^2+3x+9} = \frac{(x-3)(x+3)}{(x-3)(x^2-2x+3)}$$

and $$\lim_{x\to a}\left[\text{this}\right] = \lim_{x\to 3}\frac{x+3}{x^2-2x+3} = \text{"}\frac{6}{0}\text{"}$$

<u>diverges;</u>

- For more general functions:

L'Hôpital's rule (I): If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

(and if the latter does not exist, neither does the former)

["Why?" $f(x) \approx f(a) + f'(a)(x-a)$, and same for g ; $$\frac{\cancel{f(a)} + f'(a)(x-a)}{\cancel{g(a)} + g'(a)(x-a)}$$ $= 0$]

WRITE this!

$$= \text{"}\frac{0}{0}\text{"}$$

Example: as previous.

$$\lim_{x\to 3} \frac{x^2-9}{x^3-5x^2+3x+9}$$

$$= \lim_{x\to 3} \frac{2x}{3x^2-10x+3} = \text{"}\frac{6}{0}\text{"}$$ diverges

Ex:

$$\lim_{x\to 0} \frac{e^x-1}{x} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 0} \frac{e^x}{1} = e^0 = \underline{\underline{1}}$$

Ex:

$$\lim_{x\to 0} \frac{e^x-x-1}{x^2} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 0} \frac{e^x-1}{2x} = \text{"}\frac{0}{0}\text{"}$$ Can use l'Hôpital's rule again

$$= \underline{\underline{\tfrac{1}{2}}}$$ from previous ex.

More l'Hôpital's rule: The formula

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

holds for "$\frac{0}{0}$", but also for "$\pm \frac{\infty}{\infty}$".

Provided "$\frac{0}{0}$" or "$\pm \frac{\infty}{\infty}$": rule

- applies to $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$
- applies to $\lim_{x \to -\infty}$ or $\lim_{x \to \infty}$
- Can be adapted to

"$0 \cdot \infty$" write as "$0 \cdot \frac{1}{0}$" or "$\frac{1}{\infty} \cdot \infty$"

"1^{∞}" write as $e^{\infty \ln 1}$

i.e: For $\lim f(x)^{g(x)}$ where $f \to 1$, $g \to \infty$

Write $e^{\lim g(x) \ln f(x)}$

"0^0" write as $e^{0 \ln 0}$

"∞^0" write as $e^{0 \ln \infty}$

(But $\infty - \infty$ could be awkward!)

Example: $\lim_{x \to +\infty} x^{-1/x} = e^{\lim_{x \to +\infty} \left(-\frac{1}{x} \ln x\right)}$, $\frac{\ln x}{x} \to \frac{\infty}{\infty}$.

So $\lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$

Answer $= e^{-0} = \underline{\underline{1}}$

Important case: Let $p > 0$, $a > 0$ (constants)

Then $\lim_{x \to +\infty} \frac{x^p}{e^{ax}} = 0$. The exponential dominates polynomial growth.

Proof: it equals $\Big(\underbrace{\lim_{x \to +\infty} \frac{x}{e^{ax/p}}}_{= \frac{\infty}{\infty}}\Big)^p = \left(\lim_{x \to +\infty} \frac{1}{\frac{a}{p} e^{ax/p}}\right)^p = 0^p$

Also: For any $r > 0$, $\lim_{x \to +\infty} \frac{\ln x}{x^r} = 0$.

Proof: $\frac{\infty}{\infty}$, l'Hôpital once.

Example: $\displaystyle\lim_{x\to\infty} \frac{x^{\pi}(\ln x)^e}{x^{-2019}\, e^{x/2019}}$

Had it been "x^e" rather than $(\ln x)^e$

we would have $\displaystyle\lim_{x\to+\infty} \frac{x^{2019+\pi+e}}{e^{x/2019}} = 0$

because exponential decay kills the power.

"Fancy fix": Since $x \geq \ln x$ (all x, but what matters: large x)

$$0 \leq \frac{x^{\pi}(\ln x)^e}{x^{-2019}\, e^{x/2019}} \leq \frac{x^{\pi+e}}{x^{-2019}\, e^{x/2019}} = \frac{x^{2019+\pi+e}}{e^{x/2019}}$$

which $\to 0$. this = which $\to 0$.

Likely easier to spot: $\underbrace{\left(\frac{\ln x}{x}\right)^e}_{\to 0} \cdot \frac{x^{\pi+e}}{x^{-2019}\, e^{x/2019}}$

This line added afterwards (... the $\left(\frac{\ln x}{x}\right)^e \frac{x^{\pi+e}}{\cdots}$ means I've expanded by $\frac{x^e}{x^e}$)

[Revised - a bit more text added] cf discussion in class

One page became nearly two after writing out in more detail.
Next page bottom: I added a twist to the $\ln(3x)/\ln(5x^2)$ problem - letting $x \to 0^+$ rather than to infinity.

Example without/with l'Hôpital:

(a) Find $\lim\limits_{x \to +\infty} \dfrac{2019x^{2018} + x^{2019}}{e^{-x} + x^{2019}}$ <u>without</u> l'Hôpital.

(b) Find $\lim\limits_{x \to +\infty} \left(\dfrac{1}{x} \ln\left(1 + x^{\pm 2019} e^{x}\right)\right)$, cases i) exponent $= 2019$, ii) exponent $= -2019$

(a) The fastest-growing terms: 2019^{th} power.

Multiply by $\dfrac{x^{-2019}}{x^{-2019}}$: $\dfrac{(2019x^{2018} + x^{2019})\, x^{-2019}}{(e^{-x} + x^{2019})\, x^{-2019}} = \dfrac{\frac{2019}{x} + 1}{x^{-2019}e^{-x} + 1}$

Take $\lim\limits_{x \to +\infty}$: $\dfrac{2019}{x} \to 0$, $x^{-2019}e^{-x} \to 0$, answer: $\dfrac{0+1}{0+1} = \underline{\underline{1}}$.

(b) Case i): Since $1 + x^{2019}e^{x} \to +\infty$, $\dfrac{\ln(1 + x^{2019}e^{x})}{x} \to$ "$\dfrac{\infty}{\infty}$";

l'Hôpital: $\lim\limits_{x \to +\infty} \dfrac{\dfrac{2019x^{2018}e^{x} + x^{2019}e^{x}}{1 + x^{2019}e^{x}}}{1}$ ⟵ multiply by $\dfrac{e^{-x}}{e^{-x}}$

arriving at the limit from (a), which is $\underline{\underline{1}}$

[ex. Cont'd]

Case ii) : We need to check whether $\ln(1+x^{-2019}e^x) \to \infty$

$\iff \lim_{x\to+\infty} \frac{e^x}{x^{2019}} = +\infty$, which is true: exp growth dominates the power function.

Then we have "$\frac{\infty}{\infty}$" and can use l'Hôpital to get

$$\lim_{x\to+\infty} \frac{\frac{-2019x^{-2020}e^x + x^{-2019}e^x}{1+x^{-2019}e^x}}{1}$$

$\longleftarrow$ again, multiply this by $\frac{e^{-x}}{e^{-x}}$

these two could have been done in one operation

$$= \lim_{x\to+\infty} \frac{-2019x^{-2020} + x^{-2019}}{e^{-x} + x^{-2019}}.$$

Much like (a): multiply by $\frac{x^{2019}}{x^{2019}}$

$$= \lim_{x\to+\infty} \frac{-2019/x + 1}{x^{2019}e^{-x} + 1}.$$

Again, the exp wins: $x^{2019}e^{-x} \to 0$

$$= \frac{0+1}{0+1} = \underline{\underline{1}}.$$

Ex: In class — next page — I did $\lim_{x\to+\infty} \frac{\ln 3x}{\ln 5x^2}$. Let me do the l'Hôpital method too, but for a twist: $\lim_{x\to 0^+}$ instead. Still $\frac{\infty}{\infty}$, so:

example added later

$$\lim_{x\to 0^+} \underbrace{\frac{\ln(3x)}{\ln(5x^2)}}_{\to -\infty/-\infty} = \lim_{x\to 0^+} \frac{1/x}{10x/5x^2} = \lim_{x\to 0^+} \frac{1}{2} = \underline{\underline{\frac{1}{2}}}.$$

Ex: $\lim\limits_{x \to +\infty} \frac{\ln(3x)}{\ln(5x^2)} = \lim\limits_{x\to+\infty} \frac{\ln 3 + \ln x}{\ln 5 + 2\ln x} = \lim\limits_{x\to+\infty} \frac{\frac{\ln 3}{\ln x} + 1}{\frac{\ln 5}{\ln x} + 2} = \underline{\underline{\frac{1}{2}}}$

"Pitfalls" of l'Hôpital's rule:

- You MUST check – and on the exam claim to have checked – that you get "$\frac{0}{0}$" or $\pm$ "$\frac{\infty}{\infty}$"
- It's $\frac{f'(x)}{g'(x)}$, not $\left(\frac{f}{g}\right)'$.
- Sometimes l'Hôpital won't "help" – expressions might "get worse".
  - Sometimes you have to rewrite first and then l'Hôpital – and/or: in between successive applications

... and that "$\infty - \infty$" ... ?

$\infty - \infty$ examples

(a) $\lim_{x \to -\infty} \left(e^{-x} + x^3 + x^2 \right)$

(b) $\lim_{x \to +\infty} \left(\sqrt{x^2 + px + q} - x \right)$

(a) As $x \to -\infty$, $e^{-x} \to +\infty$ "as $\lim_{z \to +\infty} e^z$"; "fast".

— , — $x^3 + x^2 \to -\infty$ not so fast.

Write $e^{-x} \cdot \left[1 + \underbrace{\underbrace{(x^3 + x^2)}_{\to -\infty,\ \text{polynomial}} \underbrace{e^x}_{\to 0,\ \text{exponentially}}}_{\text{as } x \to -\infty,\ \to 0} \right] \Rightarrow$ "$+\infty \cdot [1 + 0]$"

largest order dominates

(b) For "such" problems, $\sqrt{a} - \sqrt{b} = \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}}$

We get $\lim_{x \to +\infty} \frac{\cancel{x^2} + px + q - \cancel{x^2}}{\sqrt{x^2 + px + q} + x} = \lim_{x \to +\infty} \frac{p + q/x}{\sqrt{1 + \frac{p}{x} + \frac{q}{x^2}} + 1} = \underline{\underline{\frac{p}{2}}}$