Integration

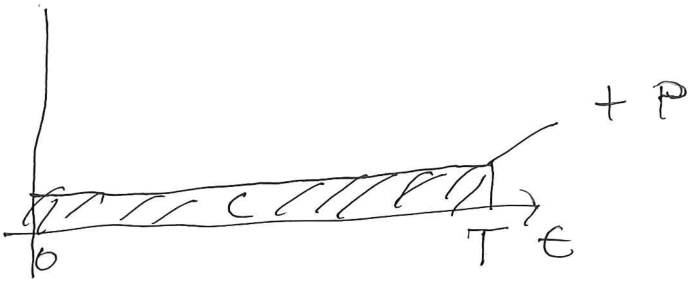
Motivation

i) Area under a curve

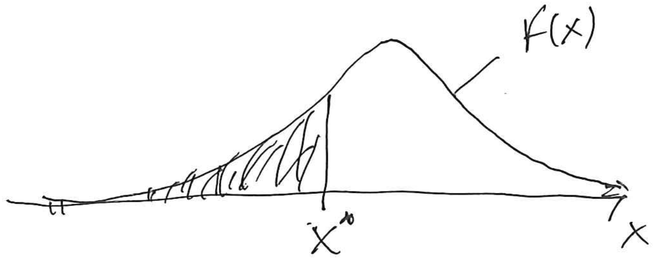
$$A = \int_a^b f(x)\,dx$$

ii) Value of a bond: $c>0$, T, $r>0$, $P>0$

$$V = \int_0^T e^{-rt} c\,dt + e^{-rT} P$$

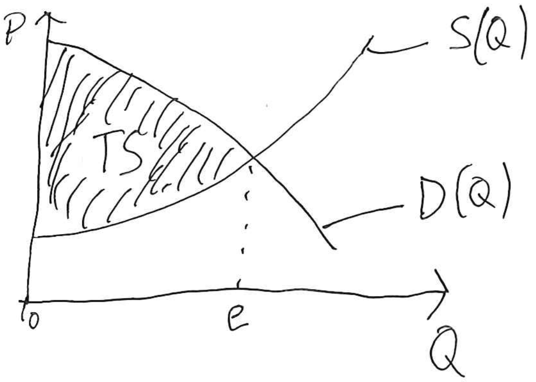


iii) Probabilities



$$\Pr(x \le x^*) = \int_{-\infty}^{x^*} f(x)\,dx$$

iv) Total Surplus



$$TS = \int_0^e \big(D(Q) - S(Q)\big)\,dQ$$

Indefinite integral

Derivative:

$$\frac{d}{dx} x^{a+1} = (a+1)x^{a}$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Antiderivative: $\underbrace{}_{}$ Indefinite integral

$$\int (a+1)x^{a}\,dx = \underbrace{x^{a+1}}_{\text{antiderivative}} + C, \quad C \in \mathbb{R}$$

$$\int a e^{ax}\,dx = e^{ax} + C$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

Why do we have $\underline{C}$? Well, $\frac{d}{dx}\left(x^{a+1} + \underline{C}\right) = (a+1)x^{a}$

Can we do the reverse operation of derivation? Can we find an anti-derivative?

$$\int x^{a+1}\,dx = \frac{1}{a+2}x^{a+2} + C, \quad \frac{d}{dx}\left(\frac{1}{a+2}x^{a+2} + C\right) = x^{a+1}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad \frac{d}{dx}\left(\frac{1}{a}e^{ax} + C\right) = e^{ax}$$

$$\int \ln x\,dx = x(\ln x - 1) + C, \quad \frac{d}{dx}\left(\underline{x}(\ln x - 1) + C\right)$$

$$= \ln x - 1 + \cancel{x}\left(\frac{1}{\cancel{x}}\right) = \ln x$$

Generally we write:

$$\int f(x)\,dx = \underbrace{F(x)}_{\text{a function}} + \underbrace{C}_{\text{a constant}}$$

"=" Class of functions

Indefinite integral, cont.

Some rules:

$a \neq -1$, then $\int x^a dx = \frac{1}{a+1} x^{a+1} + C$

$a = -1$, $x > 0$ then $\int \frac{1}{x} dx = \ln x + C$

$a \neq 0$, $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$

$$\int a f(x) dx = a \int f(x) dx$$

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

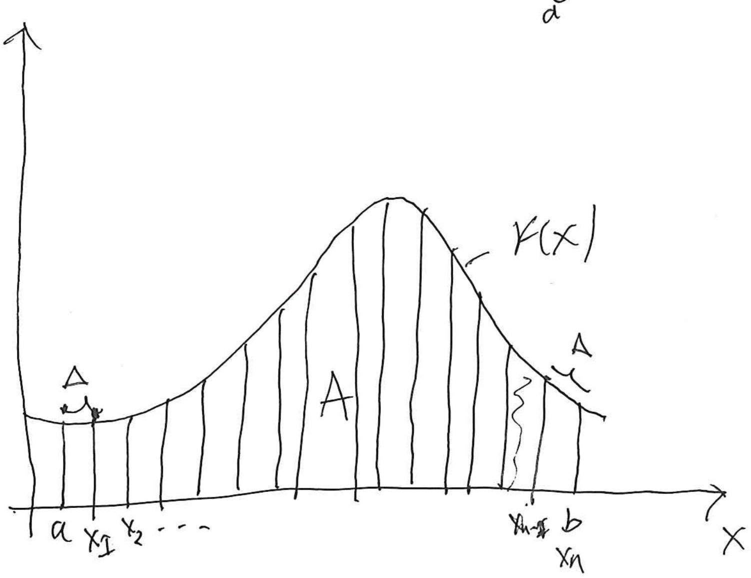
Example: $\int (x^a + b e^{cx}) dx = \int x^a dx + b \int e^{cx} dx = \frac{1}{a+1} x^{a+1} + \frac{b}{c} e^{cx} + B$

$a \neq -1$, $c \neq 0$

Definite integral

We know what $\int f(x) dx$ means. What does $\int_a^b f(x) dx$ mean?

a, b = limits of integration. $\int_a^b f(x) dx$ can be interpreted as an area under the curve $f(x)$: $A = \int_a^b f(x) dx$.



$A(a,b) =$ "area from a to b"

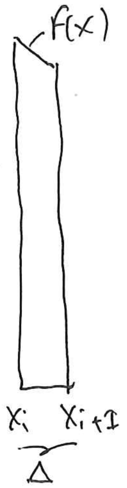
$x_0 = a$, $x_n = b$, $\Delta = \frac{b-a}{n}$

$x_i = a + i \cdot \Delta$

$$A_n = \sum_{i=1}^{n} f(x_i^*) \Delta, \quad x_i^* \in [x_{i-1}, x_i]$$

? Can we do this

Suppose $f(x_i) > f(x_{i+1})$



$$f(x_i)\cdot\Delta \geq \Delta A = A(a, x_i+\Delta) - A(a, x_i)$$
$$\geq f(\underbrace{x_i+\Delta}_{x_{i+1}})\Delta$$

Divide with Δ:

$$f(x_i) \geq \frac{A(a, x_i+\Delta) - A(a, x_i)}{\Delta} \geq f(x_i+\Delta)$$

Let $\Delta \to 0$: $\quad f(x_i) \geq A_{x_i}(a, x_i) \geq f(x_i)$

Fix a, write $A(a,b) = A(b)$ $\Rightarrow$ $f(x_i) = A'(x_i)$

The derivative of A (area) is the curve's "height", $f(x_i)$.

A must be one of $f(x_i)$'s indefinite integrals (antiderivatives)! $A'(x_i) = f(x_i)$. We want "the correct one" $A(b)$. Let $F(x) = \int f(x)dx$

$$A(x) = \underline{F(x) + C}$$

$$A(a) = 0 = F(a) + C \iff C = -F(a)$$

$$\Rightarrow \quad A(x) = F(x) - F(a)$$
$$A(b) = F(b) - F(a) = \int_a^b f(x)dx$$

$$\boxed{\text{Definite integral: } \int_a^b f(x)dx = F(b) - F(a)}$$

Often: $\int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a)$. We can have $a > b$

$$\int_b^a f(x)dx = F(a) - F(b) = -\int_a^b f(x)dx.$$

Definite integral: $\underline{\text{a number}}$

Indefinite integral: Class of functions

$$\int_1^2 (x^2-2)\,dx \,,$$

Indefinite integral: $\int (x^2-2)\,dx = \frac{1}{3}x^3 - 2x + c$

$\Rightarrow \int x^2 dx - \int 2\,dx$

$$\int_1^2 \underline{(x^2-2)}\,dx = \Big|_1^2 \underbrace{\left(\frac{1}{3}x^3 - 2x + c\right)}_{F(x)+c} = \underbrace{\frac{1}{3}2^3}_{\frac{1}{3}\cdot 8} - \underbrace{2\cdot 2}_{4} - \left(\underbrace{\frac{1}{3}\cdot 1^3}_{\frac{1}{3}} - \underbrace{2\cdot 1}_{2}\right)$$

$\Rightarrow F(2) \cancel{+c} - F(1) - \cancel{c}$

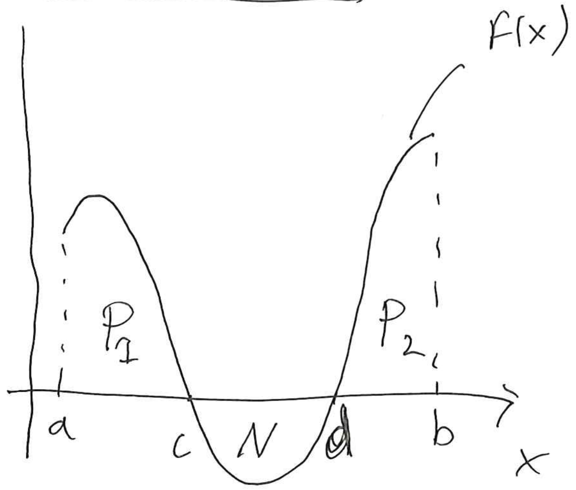
$$= \frac{1}{3}$$

$$\int_0^5 10\cdot e^{-0.1t}\,dt = \Big|_0^5 -\frac{10}{0.1} e^{-0.1t} = -\frac{10}{0.1} e^{-0.1\cdot 5} - \left(-\frac{10}{0.1}\right) e^{-0.1\cdot 0}$$

$$= -100e^{-0.5} + 100$$

$$= \underline{\underline{100(1-e^{-0.5})}} \approx 39$$

When $f(x) < 0$



$$P_1 = \int_a^c f(x)\,dx > 0$$

$$P_2 = \int_d^b f(x)\,dx > 0$$

$$N = -\int_c^d f(x)\,dx > 0$$

We have $A = \int_a^b f(x)\,dx = P_1 - N + P_2 \left(= F(b) - F(a)\right)$

Many applications have negative areas, e.g. cash flows.

Properties of definite integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx \,, \qquad \alpha \text{ is a constant}$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \qquad a < c < b$$

$$\int_a^b \left(f(x) + g(x)\right)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$