Recap:

Indefinite integrals: $\int f(x)\,dx = F(x) + C$, $\frac{d}{dx}F(x) = f(x)$

Definite integrals: $\int_a^b f(x)\,dx = \underline{F(b)} - \underline{F(a)}$

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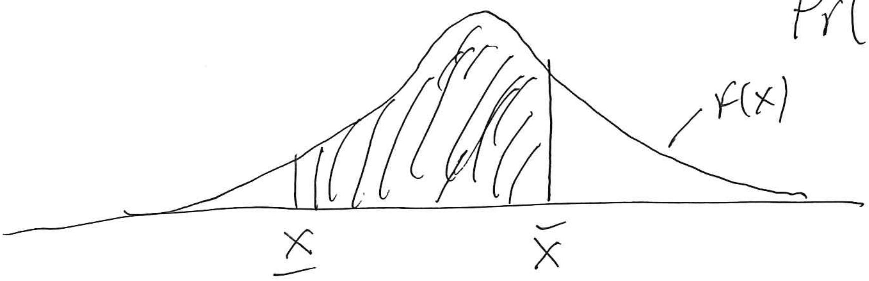
ii) Value of a bond: $c > 0$, $r > 0$, $P = 0$, T

$$V = \int_0^T e^{-rt} c\,dt = c\Big|_0^T -\frac{1}{r}e^{-rt} = c\left(-\frac{1}{r}e^{-rT} - \left(-\frac{1}{r}\underbrace{e^{-r\cdot 0}}_{=1}\right)\right)$$

$$= \underline{\underline{(1-e^{-rT})\frac{c}{r}}}$$

iii) Probability



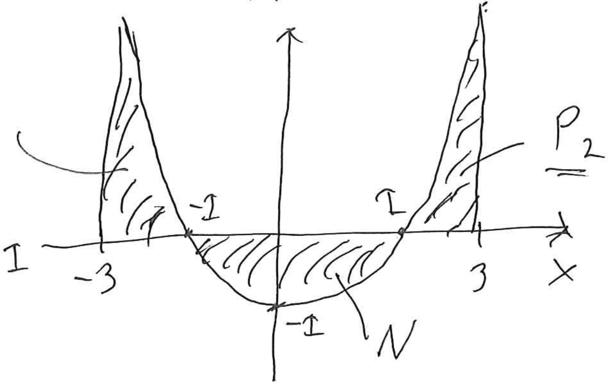
$$\Pr(\underline{x} \le x \le \bar{x}) = \int_{\underline{x}}^{\bar{x}} f(x)\,dx$$

$$= \int_{\underline{x}}^{\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

= not expressible by elementary functions!

Example: $f(x)$ is negative for some x

$f(x) = x^2 - 1$



$f(x) = 0$

$x^2 - 1 = 0 \Leftrightarrow x^2 = 1$

$A = P_1 - N + P_2$

$$A = \int_{-3}^{3} (x^2 - 1)\,dx = \Big|_{-3}^{3} \left(\tfrac{1}{3}x^3 - x\right) = \underbrace{\underbrace{\tfrac{1}{3}3^3}_{3^2=9} - 3}_{F(b)} - \underbrace{\left(\underbrace{\tfrac{1}{3}(-3)^3}_{-9} - \underbrace{(-3)}_{+3}\right)}_{F(a)}$$

$$= 9 - 3 + 9 - 3 = \underline{12}$$

$$P_1 = \int_{-3}^{-1} (x^2 - 1)\,dx = \Big|_{-3}^{-1} \left(\tfrac{1}{3}x^3 - x\right) = \underbrace{\tfrac{1}{3}(-1)^3}_{-\frac{1}{3}} - \underbrace{(-1)}_{+1} - \underbrace{\left(\tfrac{1}{3}(-3)^3 - (-3)\right)}_{-9+3}$$

$\frac{2}{3}$

$$= -\tfrac{1}{3} + 1 + 9 - 3 = 6\tfrac{2}{3}$$

$$P_2 = \int_{1}^{3} (x^2 - 1)\,dx = 6\tfrac{2}{3}$$

$$N = -\int_{-1}^{1} (x^2 - 1)\,dx = -\Big|_{-1}^{1} \left(\tfrac{1}{3}x^3 - x\right) = -\left(\tfrac{1}{3}\cdot 1^3 - 1 - \underbrace{\left(\tfrac{1}{3}(-1)^3 - (-1)\right)}_{\left(-\frac{1}{3} + 1\right)}\right)$$

$$= -\left(\tfrac{1}{3} - 1 + \tfrac{1}{3} - 1\right) = -\left(-\tfrac{2}{3} - \tfrac{2}{3}\right) = \tfrac{4}{3}$$

$$A = P_1 - N + P_2 = 6\tfrac{2}{3} - \tfrac{4}{3} + 6\tfrac{2}{3} = 12\tfrac{4}{3} - \tfrac{4}{3} = \underline{12}$$

Integration by Parts

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x) \quad \text{(Product rule)}$$

Take indefinite integrals from both sides

$$\underbrace{\int \frac{d}{dx}\left(f(x)g(x)\right)dx}_{f(x)g(x)\ (+c)} = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

Integration by Parts formula!

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

OR

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx} \quad \text{Integration by Parts}$$

Does not look very useful, but turns out to be:

- Products are difficult to integrate
- We can often rewrite $H(x)$ as $f(x)g'(x)$
- Especially useful if we get $f'(x) = 1$

Example

$$\int \underbrace{\ln x}_{f(x)}\ \overset{g'(x)}{\_}\ dx \qquad\qquad \int fg' = fg - \int f'g$$

Choose $f(x) = \ln x$ $\qquad g'(x) = 1$

$f'(x) = \frac{1}{x}$ $\qquad g(x) = x$

$$\int \ln x\,dx = (\ln x)x - \int \frac{1}{\cancel{x}}\overset{1}{\cancel{x}}\,dx = x\ln x - x + C$$

$$= x(\ln x - 1) + C \qquad \boxed{\frac{d}{dx}\left(x(\ln x - 1) + C\right) = \ln x}$$

Integration by parts: examples

$$\int f g' = fg - \int f' g$$

$$\int x e^x dx$$

Choose: $f(x) = x$ $\quad g'(x) = e^x$

$\qquad f'(x) = 1$ $\quad g(x) = e^x$

$$\int \underbrace{x}_{f} \underbrace{e^x}_{g'} dx = x e^x - \int 1 \cdot e^x dx = x e^x - e^x + C$$

$$I = \int \ln x \cdot \frac{1}{x} dx \qquad \int f g' = fg - \int f' g$$

Choose: $f(x) = \ln x$ $\quad g'(x) = \frac{1}{x}$

$\qquad f'(x) = \frac{1}{x}$ $\quad g(x) = \ln x$

$$I = \int \ln x \frac{1}{x} dx = (\ln x)(\ln x) - \underbrace{\int \frac{1}{x} \ln x \, dx}_{I}$$

$$\Rightarrow \quad I = (\ln x)^2 - I + C_I$$

$$2I = (\ln x)^2 + C_I$$

$$I = \frac{1}{2}(\ln x)^2 + C \qquad \left(C = \frac{1}{2} C_I\right)$$

Integration by Parts $\int f g' = fg - \int f' g$

Example of using int. by parts several times:

i) $\int x^3 e^x \, dx$ Choose: $f(x) = x^3 \quad g'(x) = e^x$

$f'(x) = 3x^2 \quad g(x) = e^x$

ii) $\int x^3 e^x \, dx = x^3 e^x - \int 3x^2 e^x \, dx + C_1$

? $\xrightarrow{\text{int. by Parts}}$ $f(x) = 3x^2 \quad g'(x) = e^x$

$f'(x) = 6x \quad g(x) = e^x$

iii) $\int 3x^2 e^x \, dx = 3x^2 e^x - \int 6x e^x \, dx + C_2$

? $\xrightarrow{\text{int. by Parts}}$ $f(x) = 6x \quad g'(x) = e^x$

$f'(x) = 6 \quad g(x) = e^x$

iv) $\int 6x e^x \, dx = 6x e^x - \int 6 \cdot e^x \, dx + C_3$

$= 6x e^x - 6e^x = 6(xe^x - e^x) + C_3$

iii) $\Rightarrow \int 3x^2 e^x \, dx = 3x^2 e^x - 6e^x(x-1) + C_4$

ii) $\Rightarrow \int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6e^x(x-1) + C$

C is a constant

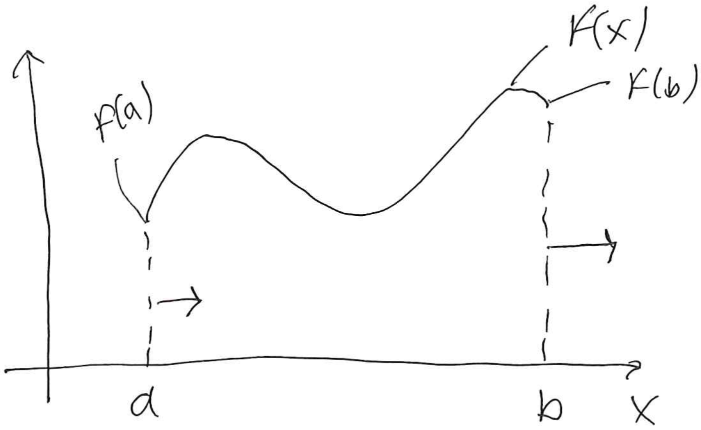
Differentiation w.r.t. limits of integration

$a =$ lower limit
$b =$ upper limit

$$\left(\frac{d}{dx}\int f(x)\,dx = f(x)\right)$$

$$\frac{d}{da}\int_a^b f(x)\,dx = \frac{d}{da}\big(F(b) - F(a)\big) = -f(a)$$

$$\frac{d}{db}\int_a^b f(x)\,dx = \frac{d}{db}\big(F(b) - F(a)\big) = f(b)$$



Increase a : $-f(a)$
Increase b : $f(b)$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x)\,dx = \frac{d}{dt}\Big(F(b(t)) - F(a(t))\Big)$$
$$= f(b(t))\,b'(t) - f(a(t))\,a'(t)$$

Example : i) $\displaystyle \frac{d}{da}\left(\int_a^b (x^2 + 2x)\,dx\right) = \overbrace{-a^2 - 2a}^{-f(a)} < 0$ $\quad a > 0$

ii) $\displaystyle \frac{d}{dt}\left(\int_0^{t^c} (x^3 - x^2)\,dx\right) =$

$$= \left((t^c)^3 - (t^c)^2\right) c\,t^{c-1}$$

Leibniz rule : $\displaystyle \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx$

Integration by Substitution

i) $\int \underbrace{(x^2+10)}_{u}{}^{50} \underbrace{2x\,dx}_{du}$

Solving this directly is practilly impossible and integration by parts fails.

What can we do? Well, we can substitute something in place of the original functions.

Let's try: $u = x^2+10$ $\quad \frac{du}{dx} = 2x \iff du = 2x\,dx$

$\Rightarrow \int u^{50}\,du = \frac{1}{51}u^{51} + C = \frac{1}{51}(x^2+10)^{51} + C$

ii) $\int \frac{1}{1+x}\,dx$ $\quad \left(\int \frac{1}{x}\,dx = \ln x + C\right)$ $\quad (x>0)$

Let's try $u = 1+x$ $\quad \frac{du}{dx} = 1 \iff du = dx$

$\Rightarrow \int \frac{1}{u}\,du = \ln u + C = \ln(1+x) + C$ $\quad \left(\begin{array}{l} 1+x>0 \\ x>-1 \end{array}\right)$