Integration by substitution, examples

i) $\displaystyle\int \frac{x^2}{1+x^3}\,dx$ $u = 1+x^3$ $du = 3x^2 dx$

$\iff dx = \frac{1}{3x^2} du$

$$= \int \frac{1}{u} \cancel{x^2} \frac{1}{3\cancel{x^2}} du = \frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|1+x^3| + C$$

ii) $\displaystyle\int \frac{x}{(1+x)^2}\,dx$ $u = (1+x)^2$ $du = 2(1+x)dx$

$\iff dx = \frac{1}{2(1+x)} du$

$$= \int \frac{1}{u} \times \frac{1}{\underbrace{2(1+x)}_{\sqrt{u}}} du = \int \frac{1}{u} \frac{x}{2\sqrt{u}} du$$

$u = (1+x)^2 \iff \sqrt{u} = 1+x \iff \underline{x = \sqrt{u} - 1}$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u} \frac{\sqrt{u}-1}{\sqrt{u}} du = \frac{1}{2}\int \frac{1}{u}\left(1 - \frac{1}{\sqrt{u}}\right) du \qquad \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(\int \frac{1}{u} du - \int \frac{1}{u^{\frac{3}{2}}} du\right) = \frac{1}{2}\left(\ln u - \left(\frac{1}{-\frac{3}{2}+1}\right) u^{-\frac{1}{2}} + \hat{C}\right)$$

$$= \frac{1}{2}\left(\ln u + 2u^{-\frac{1}{2}}\right) + C = \frac{1}{2}\left(\ln(1+x)^2 + 2\frac{1}{1+x}\right) + C = \underline{\ln(1+x) + \frac{1}{1+x} + C}$$

Integration by substitution

How to find $\int G(x)dx$?

1. Pick out a "part" of $G(x)$ as a new variable $u = g(x)$
2. $du = g'(x)dx$ and find $dx = \frac{du}{g'(x)}$
3. Substitute $u = g(x)$ and $dx = \frac{du}{g'(x)}$ to get from $\int G(x)dx$ to $\int f(u)du$
4. Find $\int f(u)du = F(u) + C$
5. Replace u by $g(x)$ to get $F(g(x)) + C$

Example: $\int x^3 \sqrt{1+x^2}\, dx$

1. $u = \sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}}$
2. $du = \frac{1}{2}\underbrace{(1+x^2)^{-\frac{1}{2}}}_{\frac{1}{u}} 2x\, dx \iff u\,du = x\,dx$
3. $(u^2-1)u = (1+x^2-1)\sqrt{1+x^2} = x^2\sqrt{1+x^2}$

$$\int \underbrace{x^2\sqrt{1+x^2}}_{(u^2-1)u} \cdot \underbrace{x\,dx}_{u\,du} = \int (u^2-1)u^2 du = \int (u^4-u^2)du$$

4. $= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$
5. $\int x^3\sqrt{1+x^2}dx = \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 + C$

Substituting more than once

Example: $\int \frac{1}{x \ln(x) \ln(\ln(x))} dx$, $\quad x > 0$

$V = \ln(x) \qquad dv = \frac{1}{x} dx$

$$\int \frac{1}{\ln(x) \ln(\ln(x))} \frac{1}{x} dx = \int \frac{1}{V} \frac{1}{\ln(V)} dV$$

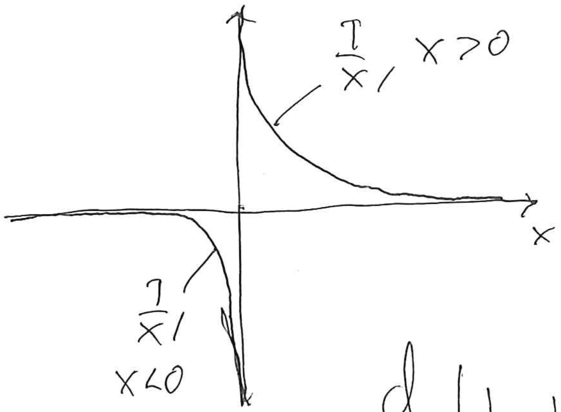
$u = \ln(v) \qquad du = \frac{1}{v} dv$

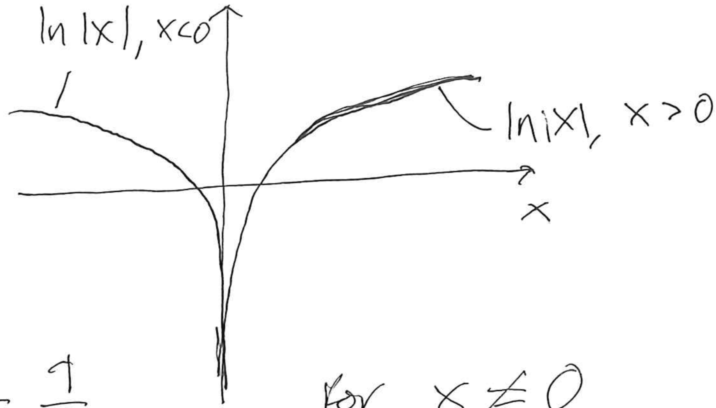
$$\int \frac{1}{\ln(v)} \frac{1}{v} dv = \int \frac{1}{u} du = \ln(u) + C$$
$$= \ln(\ln(v)) + C$$
$$= \ln(\ln(\ln(x))) + C$$

Domains of functions and integration

So far $\int \frac{1}{x}\,dx = \ln x$, works for $x>0$, i.e. we restrict the domain of the function.

However, $\frac{1}{x}$ is defined for all $x \neq 0$ $(x \in \mathbb{R} \setminus \{0\})$





$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \text{for } x \neq 0$$

Thus, $\int \frac{1}{x}\,dx = \ln|x|$

What about $\int_a^b \frac{1}{x}\,dx$? If $a, b > 0$ or $a, b < 0$, no problems.

If $a<0$, $b>0$, we are integrating over a "jump", at which the integrand tends to $\pm$ infinity.

Solution: we define the integral as $\lim_{c \to 0} \int_a^c \frac{1}{x}\,dx + \lim_{d \to 0} \int_d^b \frac{1}{x}\,dx$

But what is the value of $\lim_{d \to 0} \int_d^b \frac{1}{x}\,dx$?

$$\lim_{d \to 0} \int_d^b \frac{1}{x}\,dx = \lim_{d\to 0} \Big|_d^b \frac{1}{x} = \lim_{d \to 0}\left(\ln(b) - \ln(d)\right) = \ln(b) - \lim_{d\to 0} \ln(d)$$

$$= \ln(b) - (-\infty) = \infty \quad \text{(diverges!)}$$

$\Rightarrow$ Integrals of unbounded functions, next

Integrals of unbounded functions

Consider $\int_0^2 \frac{1}{\sqrt{x}}\,dx$, $x \in (0,2]$ $\left(\text{Counterexample } \int_0^2 \frac{1}{x}dx\right)$

We have that $\frac{1}{\sqrt{x}} \to \infty$ as $x \to 0^+$ (Not defined at $x=0$)

We can let $\int_0^2 \frac{1}{\sqrt{x}}dx = \lim_{h\to 0^+} \int_h^2 \frac{1}{\sqrt{x}}dx = \lim_{h\to0^+} \Big|_h^2 2\sqrt{x}$

$= \lim_{h\to0^+} (2\sqrt{2} - 2\sqrt{h}) = 2\sqrt{2}$, Converges!

More generally: let $f(x)$ be continuous on $(a, b]$, then

$$\int_a^b f(x)dx = \lim_{h\to0^+} \int_{a+h}^b f(x)dx$$

If this limit exists, the integral converges and is well-defined.

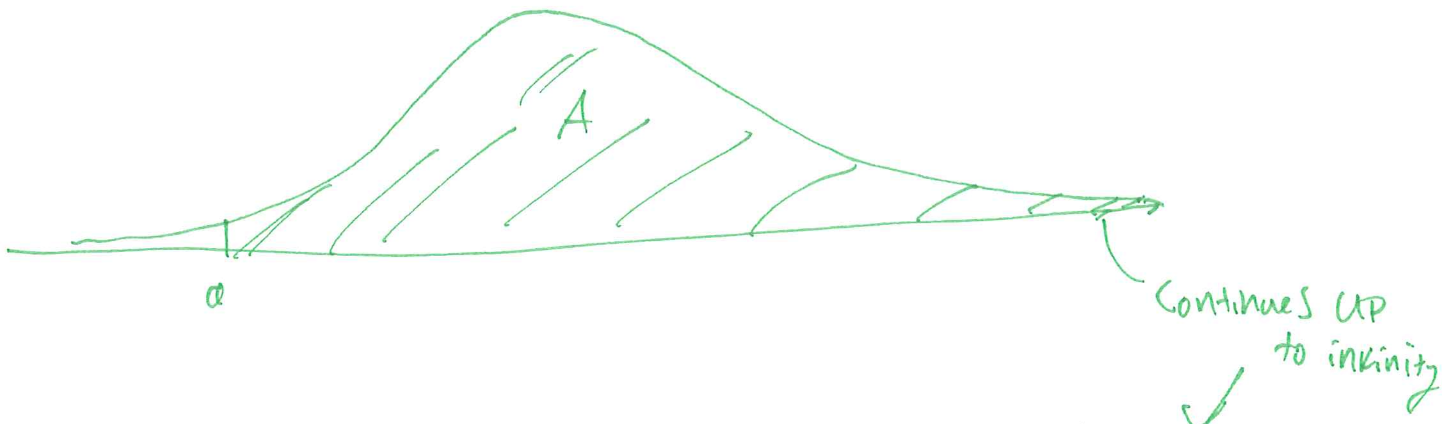
The cases when $f(x)$ is continuous on $[a,b)$ and continuous (a,b) are analoguous:

$$\int_a^b f(x)dx = \lim_{h\to0^+} \int_{a+h}^{b-h} f(x)dx.$$

If the limit exists, the integral converges.

Infinite intervals of integration

Suppose we are integrating from a to ∞. Are we allowed to do this?



Is A finite?

Contrast: $\sum_{i=1}^{\infty} a_i < \infty$? $a_i = \frac{1}{n^2}$ finite

$a_i = \frac{1}{n}$ infinite.

Formally we write the integral from a to ∞ as

$$\lim_{b \to \infty} \int_a^b f(x)\,dx \qquad (f(x) \text{ is integrable})$$

The question is does the integral above converge? E.g.:

$$\lim_{b \to \infty} \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \underbrace{\lim_{b \to \infty} \int_c^b f(x)\,dx}_{< \varepsilon ?}$$

Is there a c such? for all $\varepsilon > 0$

If the limit does not exist $\left(\lim_{b \to \infty} \int_a^b f(x)\,dx = \pm\infty\right)$

we say the integral diverges.

Note we can as well have $\lim_{a \to -\infty} \int_a^b f(x)\,dx$

or $\left(\lim_{a \to -\infty,\, b \to \infty} \int_a^b f(x)\,dx\right)$. These are often written as

$$\int_{-\infty}^b f(x)\,dx, \quad \int_{-\infty}^{\infty} f(x)\,dx, \quad \int_a^{\infty} f(x)\,dx.$$

Ex. a) $\int_0^\infty \lambda e^{-\lambda x}\,dx \quad , \lambda > 0$

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\int_0^b \lambda e^{-\lambda x}\,dx$$

$$\int \lambda e^{-\lambda x}\,dx = -e^{-\lambda x} + C$$

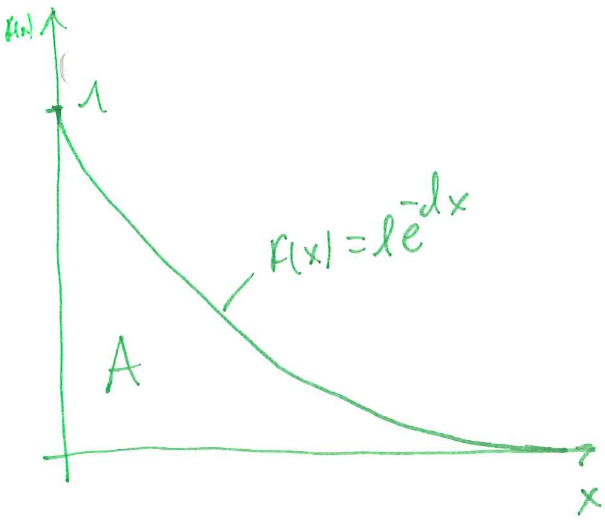
$$\int_0^b \lambda e^{-\lambda x}\,dx = \Big|_0^b \left(-e^{-\lambda x}\right) = -e^{-\lambda b} + 1$$

$$\lim_{b\to\infty}\int_0^b \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} -e^{-\lambda b} + 1 = 1,$$ Converges!

Ex. b) $\int_1^\infty \frac{1}{x}\,dx$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\Big|_1^b \ln(x)\right) = \lim_{b\to\infty}\left(\ln(b) - 0\right)$$

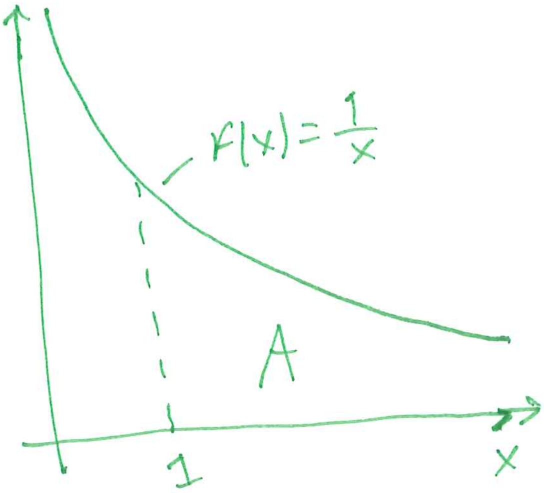
$$= \lim_{b\to\infty} \ln(b) = \infty$$



f(x) approaches 0 rapidly enough

$\lambda = 0.5 \quad x = 20$

$f(20) = 0.00007$



f(x) approaches 0 too slowly.

$f(20) = 0.05$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx$$

If both integrals converge then the improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges.

Ex.

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx = \int_{0}^{\infty} \frac{x}{1+x^2}dx + \int_{-\infty}^{0} \frac{x}{1+x^2}dx$$

use substitution: $u = 1+x^2$ $du = 2x\,dx \Leftrightarrow \frac{1}{2}du = x\,dx$

$$\int_{0}^{\infty} \frac{x}{1+x^2}dx = \lim_{b\to\infty} \int_{0}^{b} \frac{x}{1+x^2}dx = \lim_{b\to\infty} \int_{0}^{b} \frac{1}{2}\frac{1}{u}du = \lim_{b\to\infty} \left|_{0}^{b} \frac{1}{2}\ln(1+x^2)\right.$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln(1+b^2) = \infty$$ So integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx$ diverges.

(irrespective of $\int_{-\infty}^{0} \frac{x}{1+x^2}dx$)

Leibniz rule

Let $a = a(x)$, $b = b(x)$, $f(x,t)$

$$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$$

$$\boxed{F'(x) = f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt}$$ Leibniz rule

Why: $F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt = G(x,b(x)) - G(x,a(x))$

Using the chain rule

$$\frac{dF}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial b}\frac{\partial b}{\partial x} - \frac{\partial G}{\partial x} - \frac{\partial G}{\partial a}\frac{\partial a}{\partial x}$$

$$\Rightarrow \frac{dF}{dx} = f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x) + G_1(x,b(x)) - G_1(x,a(x))$$

$$\int_{a(x)}^{b(x)} f(x,t)\,dt = G(x,b(x)) - G(x,a(x))$$

$$\frac{\partial}{\partial x}\int_{a^*}^{b^*} f(x,t)\,dt = \frac{\partial}{\partial x}\left(G(x,b^*) - G(x,a^*)\right) = G_1(x,b) - G_1(x,a)$$

Thus, $G_1(x,b(x)) - G_1(x,a(x)) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$

Integration by parts and integration by substitution: <u>definite integrals</u>

i) You can always first solve the indefinite integral and then use that to solve the definite integral.

ii) Use int. by parts or substitution directly. Please see online notes.