Leibniz' rule: an example

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2}\frac{e^{tx}}{t}\,dt = \frac{e^{x^2\cdot x}}{x^2}\cdot 2x - \frac{e^{\sqrt{x}\cdot x}}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$$

$$+ \underbrace{\int_{\sqrt{x}}^{x^2}\frac{\cancel{t}e^{tx}}{\cancel{t}}\,dt}$$

$$= \left[\frac{1}{x}e^{tx}\right]_{t=\sqrt{x}}^{t=x^2}$$

We get:

$$2\frac{e^{x^3}}{x} - \frac{1}{2}\frac{e^{x^{3/2}}}{x} + \frac{1}{x}e^{x^3} - \frac{1}{x}e^{x^{3/2}}$$

$$= \frac{1}{x}\left[3e^{x^3} - \frac{3}{2}e^{x^{3/2}}\right]$$

Differential equations

what? Equation for an unknown <u>function</u> x involving some derivative(s) of x.

why? E.g.: the change in <u>time</u> t of $x = x(t)$, may depend on its current state.

Ex:

$$\frac{dx(t)}{dt} = r\,x(t) + w(t) - k \qquad (1)$$

rate of change — interest rate — current endowment level — (wage) income — consumption

Ex:

$$\frac{dx(t)}{dt} = r\,x(t)\cdot\left(1 - \frac{x(t)}{K}\right) \qquad (2)$$

"Logistic growth". Population in an environment with (constant) carrying capacity K.

Terminology:

- **Particular solution**: some function x that satisfies the diff. eq.
- **General solution**: the set of all particular solutions.

Ex: $\frac{dx}{dt} = rx$: particular solutions e^{rt}, $54e^{rt}$, ...

Turns out: Ce^{rt} is the general solution. C an arbitrary constant.

Curriculum:

highest-order derivative involved

as opposed to "partial".

i.e.: $x = x(t)$, one variable t.

For first-order ordinary differential eq's:

- Verify any particular solution
- Solve - for general and particular solution(s) -
  - → Linear : $\frac{dx}{dt} + a(t)x = b(t)$
  - → Separable : $\frac{dx}{dt} = f(t)\,g(x)$.

and eq's that can be rewritten into such.

Notation: $\dot{x}$ for $\frac{dx}{dt}$. (For time-derivative, since Newton.)

Drop "(t)" from x and $\dot{x}$.

Linear diff. eq's: $\dot{x} + a(t)\,x = b(t)$.

- Method (leads to formula, both available)

Multiply by $e^{A(t)}$ where $\dot{A}(t) = a(t)$

$$\underbrace{(\dot{x} + a(t)\,x)\,e^{A(t)}}_{= \frac{d}{dt}\left(x\,e^{A(t)}\right)} = e^{A(t)}\,b(t)$$

So $x\,e^{A(t)} = \int e^{A(t)}\,b(t)\,dt$

$\Rightarrow$

- Formula:

$$x = \underbrace{C e^{-A(t)}}_{} + \underbrace{e^{-A(t)}\int e^{A(t)}\,b(t)\,dt}_{}$$

Writing "C" explicitly:

the general solution of $\underbrace{\dot{x} + a(t)\,x = 0}_{}$

"the corresponding homogeneous eq."

Can take this as an arbitrary particular solution.

Example: $\dot{x} = rx - k$ (① with $w \equiv 0$)

$r \neq 0$ const., k const.

Find the general solution.

$a = -r$ is constant. $b = -k$ is constant.

$$x = Ce^{-A(t)} + e^{-A(t)} \int b(t) e^{A(t)} \, dt$$

choose $A = at = -rt$

$$x = Ce^{rt} + e^{rt} \int \underbrace{(-k) \cdot e^{-rt}}_{\text{one antiderivative: } \frac{k}{r} e^{-rt}} \, dt$$

$$= \underline{\underline{Ce^{rt} + \frac{k}{r}}}.$$

$\Rightarrow$ Observe: If $a \neq 0$, b are constant, a particular solution of $\dot{x} + ax = b$, is $x \equiv \frac{b}{a}$.

Example: Same diff. eq.' but: Find the particular solution s.t $x(2019) = 100\,000$.

... put $x = 100\,000$, $t = 2019$, solve for C.

Could instead: Use $A = -r(t-2019)$

$$x = C_1 e^{r\cdot(t-2019)} + e^{r(t-2019)} \int (-k)\, e^{-r(t-2019)}\, dt$$

$$= \ldots = C_1 e^{r(t-2019)} + \frac{k}{r}$$

Then solve for C_1:

$$100\,000 = C_1 e^0 + \frac{k}{r}$$

$$x = \left(100\,000 - \frac{k}{r}\right) e^{r(t-2019)} + \frac{k}{r}$$

or if you like:

$$= 100\,000\, e^{r(t-2019)} - \frac{k}{r}\left(e^{r(t-2019)} - 1\right)$$

What if: $x(2019) = x_{2019}$, just a constant?

$$x = x_{2019}\, e^{r(t-2019)} - \frac{k}{r}\left(e^{r(t-2019)} - 1\right).$$

Ex.: Find the solution of

$$\dot{x} = rx + w_0 e^{h\cdot(t-t_0)} - k, \quad \text{s.t.} \quad x(t_0) = x_0$$

(r, w_0, t_0, x_0, h, k positive constants, $h \neq r$).

Solution: Choose $A(t) = -r\cdot(t-t_0)$.

$$\frac{d}{dt}\left(x e^{A(t)}\right) = (\dot{x} - rx)\, e^{A(t)}$$

$$= e^{A(t)}\cdot\left(w_0 e^{h\cdot(t-t_0)} - k\right)$$

An antiderivative of $\frac{d}{dt}\left(x e^{A(t)}\right)$ is:

$$x(t)\, e^{-r(t-t_0)} - x_0,$$

Now, $$\int e^{-r(t-t_0)}\left(w_0 e^{h(t-t_0)} - k\right) dt$$

$$= w_0 \int e^{(h-r)(t-t_0)}\, dt - k\int e^{-r(t-t_0)}\, dt$$

$$= \frac{w_0}{h-r} e^{(h-r)(t-t_0)} + \frac{k}{r} e^{-r(t-t_0)} + Q$$

and with Q set to match $\underbrace{x(t)\, e^{-r(t-t_0)} - x_0}_{=0}$.

Put $t = t_0$: Then this $= 0$.

So $$Q = -\frac{w_0}{h-r}\cdot\cancel{e^{0}} - \frac{k}{r}\cancel{e^{0}}.$$

Solution: $$x(t) = e^{r(t-t_0)}\left[x_0 + \frac{w_0}{h-r}\left(e^{(h-r)(t-t_0)} - 1\right) + \frac{k}{r}\left(e^{-r(t-t_0)} - 1\right]\right.$$

or if you like: $$x_0 e^{r(t-t_0)} + w_0 \frac{e^{h(t-t_0)} - e^{r(t-t_0)}}{h-r} + \frac{k}{r}\left(1 - e^{r(t-t_0)}\right)$$

Alternatively: Formula, with $A = -rt$.

$$x = Ce^{rt} + e^{rt}\int \left(w_0 e^{h(t-t_0)} - k\right) e^{-rt}\, dt$$

$$= Ce^{rt} + e^{rt}\left(\int w_0 e^{-ht_0} \cdot e^{(h-r)t}\, dt - k\int e^{-rt}\, dt\right)$$

$$= Ce^{rt} + e^{rt}\left(\frac{w_0 e^{-ht_0}}{h-r} e^{(h-r)t} + \frac{k}{r} e^{-rt}\right)$$

$$= Ce^{rt} + \frac{w_0}{h-r} e^{h(t-t_0)} + \frac{k}{r}.$$

Then at t_0:

$$x_0 = Ce^{rt_0} + \frac{k}{r} + \frac{w_0}{h-r}.$$

and $$C = e^{-rt_0}\left(x_0 - \frac{k}{r} - \frac{w_0}{h-r}\right)$$

Example with $a(t)$ not constant

$$\dot{x} + 2tx = t^3 .$$ Find the general solution.

$$a(t) = 2t , \quad A(t) = t^2$$

Formula: $$x(t) = Ce^{-t^2} + e^{-t^2} \int t^3 e^{t^2} dt$$

$$\int t^3 e^{t^2} dt = \frac{1}{2} \int t \, e^{t^2} 2t dt$$

$$u = t^2 , \quad du = 2t dt$$

$$= \frac{1}{2} \int u \underset{v'}{e^u} du$$

$$= \frac{1}{2} \left[u e^u - \int e^u du \right]$$

$$= \frac{1}{2} (u-1) e^u + \text{constant.}$$

$$u = t^2 .$$

$$x(t) = Ce^{-t^2} + \frac{1}{2} e^{-t^2} (t^2 - 1) e^{t^2}$$

$$= \underline{\underline{Ce^{-t^2} + \frac{1}{2}(t^2 - 1)}}$$

Separable differential eq.'s.

Form: $\dot{x} = f(t)\, g(x)$

$f(t)$: t only; $g(x)$: x only

or, example $\dot{x} = e^{t+x}$ because $e^{t+x} = e^t e^x$.

Method: (i) Any zero z of g yields a constant particular sol'n $x(t) \equiv z$

(ii) For $g(x) \neq 0$:

$\frac{\dot{x}}{g(x)} = f(t)$, i.e: separate:

$\frac{dx}{g(x)} = f(t)\, dt$, integrate

$\int \frac{dx}{g(x)} = \int f(t)\, dt$, yields the

form: $H(x) = F(t) + C$ where

$H' = \frac{1}{g}$, $F' = f$

General solution: Gather any/all constant sol'ns from (i), and from (ii): what you get from solving $H(x) = F(t) + C$ for x.

· Particular solution, say with $x(t_0) = x_0$:

(i) If $g(x_0) = 0$: $x(t) \equiv x_0$.

(ii) Otherwise: When at

$$H(x) = F(t) + C,$$ get rid of C:

$$H(x_0) = F(t_0) + C$$

Then solve out x from

$$H(x) = F(t) - F(t_0) + H(x_0)$$

Example: general solution of (it is Ce^{rt})

$$\dot{x} = r x$$

& $f(t) = r$, $g(x) = x$.

(i) Constant solution for $x \equiv 0$.

(ii) For $x \neq 0$: $\int \frac{dx}{x} = \int r\, dt = rt + \text{constant}.$

So $\ln|x| = rt + D$

$$|x| = \underbrace{e^{D}}_{\text{arbitrary positive constant}} \cdot e^{rt}$$

$$x = Q \cdot e^{rt}$$

$Q = \pm e^{D}$ = arbitrary nonzero constant.

General solution: $\underline{\underline{x = C e^{rt}}}$

($C = 0$ from (i))

Use the fact $\frac{1}{x(1-x/k)} = \frac{1}{x} + \frac{1}{k-x}$ to:

Example: Find the general solution of

$$\dot{x} = rx\left(1 - x/k\right), \quad k>0,\ r>0 \text{ constants.}$$

("(2)" from the first diff. eq. sheet.)

Constant solutions: $x \equiv 0$, $x \equiv k$.

Otherwise: Separate:

$$\frac{dx}{x(1-x/k)} = r\,dt.$$

Integrate: $\int \left(\frac{1}{x} + \frac{1}{k-x}\right)dx = \int r\,dt$

$$\underbrace{\ln|x| - \ln|k-x|}_{\ln\left|\frac{x}{k-x}\right|} = C + rt.$$

So $\frac{x}{k-x} = \pm e^{C} e^{rt}$ $\qquad Q = \pm e^{C} \neq 0$

Solve for x: [...]

$$x = k \cdot \frac{Q}{Q + e^{-rt}}$$

General solution: $Q \in \mathbb{R}$

$$\boxed{x = k \cdot \frac{Q}{Q+e^{-rt}} \quad , \quad x = k}$$

or:

$$\boxed{x = k \cdot \frac{1}{1+Re^{-rt}}, \quad R \in \mathbb{R} \quad \text{or } x=0.}$$

Example: Find the particular solution of

$$\dot{x} = \underbrace{e^{t+x}}_{} \; , \text{ s. t. } \; x(-1) = 7$$

$$= e^{t} \cdot e^{x}, \text{ separable.}$$

$$e^{-x}\,dx = e^{t}\,dt, \text{ so}$$

$$-e^{-x} = C + e^{t}.$$

Eliminating C: $x = 7$, $t = -1$

$$C = -e^{-7} - e^{-1}$$

$$\ln e^{-x} = \ln\left(e^{-7} + e^{-1} - e^{t}\right)$$

$$\underline{\underline{x = -\ln\left(e^{-7} + e^{-1} - e^{t}\right)}}$$

(Note: $\underbrace{e^{-7} + e^{-1} - e^{t}}_{} \to 0$ as $t \to \underbrace{\ln\left(e^{-7} + e^{-1}\right)}_{T}$

$x(t)$

T

Under linear diff. eq's, we had the example

$$\dot{x} + 2tx = t^3 .$$

Question form could have been, e.g.:

(a) Show that $\frac{1}{2}(t^2-1)$ is a particular solution

(b) Find the general solution.

· Could of course have given the general solution (cf yesterday) and pointed out that $\frac{1}{2}(t^2-1)$ is one particular.

· Alternative:

(a) Calculate $\frac{d}{dt}\frac{1}{2}(t^2-1) + 2t\cdot\frac{1}{2}(t^2-1)$ and verify that you get t^3.

(b) Solve $\dot{x} + 2tx = 0$. Both linear and separable.

Solved as separable: $x \equiv 0$ or

$$\frac{dx}{x} = -2t\,dt, \quad \ln|x| = Q - t^2$$

$$\pm|x| = \underbrace{\pm e^Q}_{C} e^{-t^2}$$

C; $C=0 \leftrightarrow$ const. sol'n $x \equiv 0$.

General sol'n: $Ce^{-t^2} + \frac{1}{2}(t^2-1)$

"Odds and ends" for question types.

Ex: Find a particular solution of

$$\dot{x} = \frac{f(t)}{f(x)}$$

Question means: "of your choice".

Solution: $x(t) = t$.

Ex: x satisfies

$$\dot{x} = (1+x^2)\, e^t.$$

$\dot{x} > 0$, strictly monotone.

Show that x is an invertible function.

Ex: x satisfies $\dot{x} = g(x) \cdot (t - Q)$ ← given constant.

(a) Find a stationary point $\dot{x}(Q) = 0$. $\underline{t = Q}$.

(b) Classify it, if possible.

If $g(Q) > 0$: loc. min

If $g(Q) < 0$: loc max.

If $g(Q) = 0$: constant solution $x \equiv Q$.

Graphical representation:

Directional diagrams

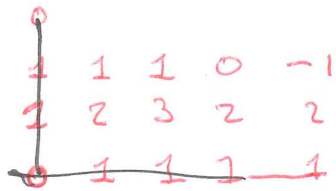
$\dot{x} = F(t, x)$.

At a given point in the (t, x) plane
– say: $(t, x) = (2, 3)$ –
$F(t, x)$ will be the slope of x
[provided we are considering the particular solution that passes through this point]

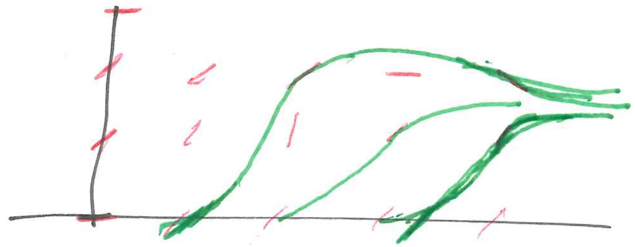
Say, if $F(2,3) = -1/2$:

"Ticking off in the tangent direction" yields
a rough diagram over directions and thus the solution curves.

Say:

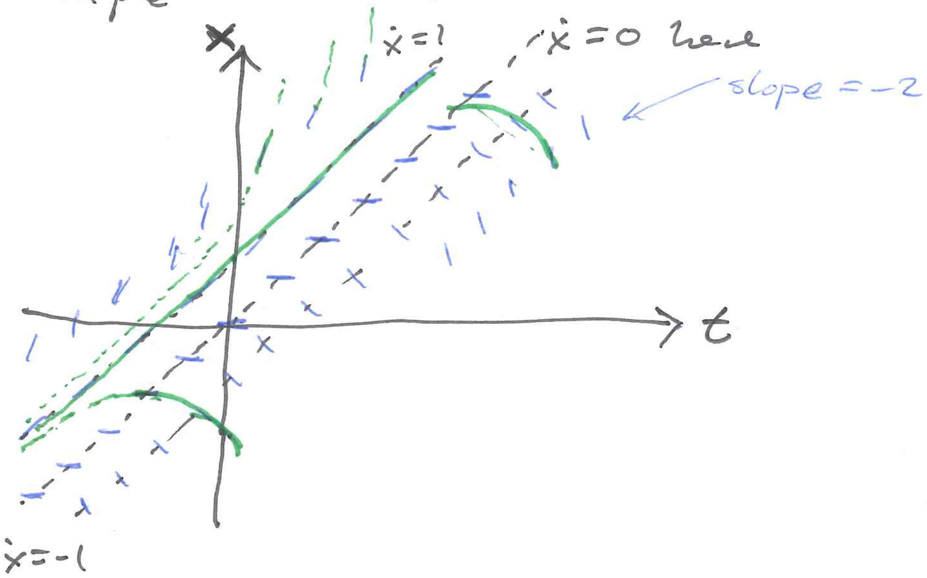


$\Longleftrightarrow$



(cannot cross!)

Example : $\dot{x} = x - t$



Remark added: The tool "phase diagram" is not Math 2 relevant. (It is a tool for $\begin{matrix} \dot{x} = F(x,y) \\ \dot{y} = G(x,y) \end{matrix}$)

Example: $\dot{x} = \underbrace{\log_2 x}_{} - \frac{1}{2}t$, for $x > 0$, $t > 0$

not so good exam problem, requires values for $\log_2 x$

Some slopes:

slope $= 0$ when $\log_2 x = \frac{1}{2}t \Leftrightarrow x = 2^{\frac{1}{2}t}$

slope $= S$ when $\log_2 x = S + \frac{1}{2}t \Leftrightarrow x = 2^{S + \frac{1}{2}t} = 2^S \cdot 2^{\frac{1}{2}t}$