LINEAR ALGEBRA NOTES

At least for the early linear algebra lectures, both slides and handwriting have been used. These notes first have the handwriting, that refer to the slideset. Opposite as of lectures.

#1: Vectors
(including the beginning of lecture 2)

- this page
- pp 2 to 5: handwriting
- pp 6 to 15: slides

LINEAR ALGEBRA

Vectors & matrices defined: see slides.

Scaling, and addition of two n-vectors:

Element-wise. Nice rules (see slides)

Dot product, two n-vectors:

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n$$

Some rules "nice":

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

$$\alpha \beta \quad \vec{x} \cdot \vec{y}$$

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot z$$

$$\vec{x} \cdot \vec{0} = 0$$

$$\vec{x} \cdot \vec{x} > 0 \quad \text{except } \vec{x} = \vec{0}$$

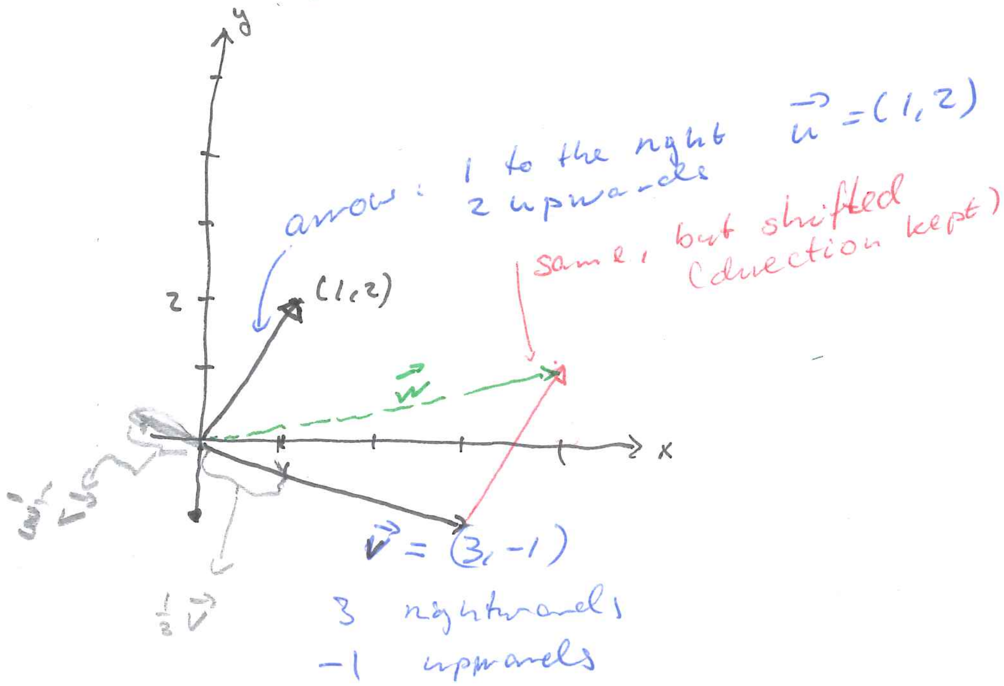
Some catches: $\vec{x} \cdot \vec{y} = 0 \not\Rightarrow$ any zero element.

Never divide by a vector

$$\underbrace{(\vec{x} \cdot \vec{y})}_{\text{number to scale } \vec{z} \text{ by}} \vec{z} \neq \vec{x} \, \underbrace{(\vec{y} \cdot \vec{z})}_{\text{number to scale } \vec{x} \text{ by}}$$

(except by coincidence)

Geometric interpretation, $n = 2$ (& 3 ... idea generalizes)



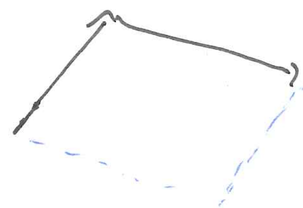
The green $\vec{w}$

:

$(1,2) + (3,-1) = (4,1)$

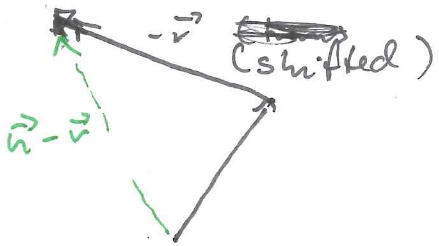
$\vec{u} + \vec{v} = \vec{w}$

$=$



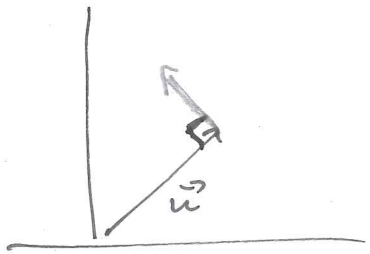
as well

Q: $\vec{u} - \vec{v}$, same $\vec{u}$ and $\vec{v}$ as above?



$\vec{u} - \vec{v} = (-2,3)$

More geometry:



Find a vector $\vec{w} \neq \vec{0}$ s.t

$\vec{u} \cdot \vec{w} = 0$ "orthogonal"

Answer: 90 degrees on $\vec{u}$

anything

Lines:

A line through the origin:

$\vec{x} = t\,\vec{u}$, $t \in \mathbb{R}$

any point on the line, except $\vec{0}$

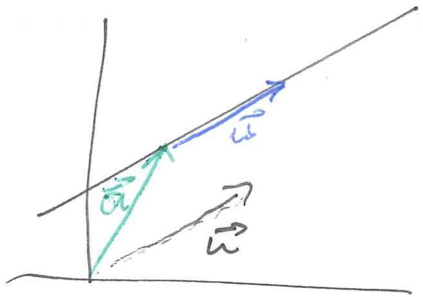
Works in $\mathbb{R}^n$

Not through the origin:

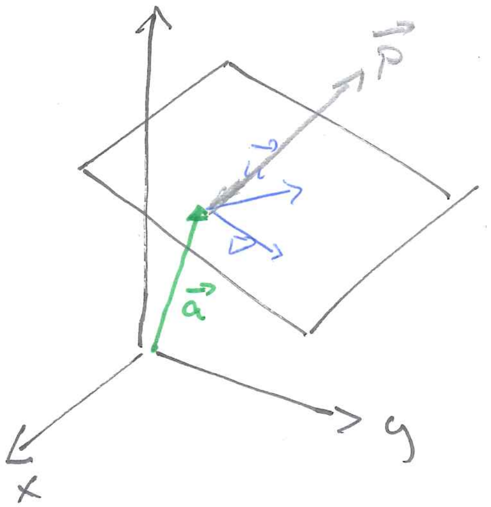
$\vec{x} = \vec{a} + t\,\vec{u}$

$\vec{a}$: point on the line

$\vec{u} \neq \vec{0}$, parallel to the line ("along the line" when shifted)



Planes in $\mathbb{R}^3$:



$\vec{a}$: some (given) point in the plane

$\vec{u}, \vec{v}$: along / parallel, <u>not</u> proportional

Any point $\vec{x}$ in the plane can be written $\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$, some $s \in \mathbb{R}$, $t \in \mathbb{R}$

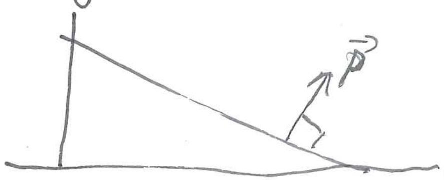
(n-1)-dimensional hyperplanes in $\mathbb{R}^n$:

Eg lines in $\mathbb{R}^2$, planes in $\mathbb{R}^3$

- In the sketch above, there is one direction $\vec{p}$ orthogonal to the plane, i.e. to both $\vec{u}$ and $\vec{v}$

$$\vec{p} \cdot \vec{x} = \vec{p} \cdot a + s\,\cancel{\vec{p} \cdot \vec{u}} + t\,\cancel{\vec{p} \cdot \vec{v}} = \vec{p} \cdot \vec{a}$$
$$= m$$

- Budget line in $\mathbb{R}^2$:



2019 linear algebra slides #1

Math 2 autumn 2018 had two auditoriums with different docucam facilities, so slides were used.

These should not be taken as "lecture notes"; they are intended to

- give the brief ideas and loose talk: an idea of where we are going to *start out.*
- cover details that will be "said once and then done automatically" and with little need to repeat before the exam.
  - Example: you will hopefully accept that the pair of numbers $(1,\ 3)$ is not the same as $(3,1)$, and you will hopefully not need to repeat it.
- and possibly: help you decide what to take notes on. (Some will indeed write down what the "$=$" sign is used for. Up to you!)

Notes intended to start from scratch and stop before the dirty work begins. (And as always: *do problems.*)

LA preview 1: Matrices and vectors – the "loose talk" of it

What they are:

- **Matrices:** Rectangular arrays ("boxes") filled with numbers.

$$\begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$$

  (Example of *order* 2×3, where *element* $(2, 1)$ equals $2e$.)

- **Vectors:** The special case of a single *row* or *column.*
  Example (row of order 3, "3-vector"): $\begin{pmatrix} 2e & -1.4 & 5 \end{pmatrix}$
  (... often comma-delimited, beware: $(2e,\ -1.4,\ 5)$)
  Example (column of order 2, "2-vector"): $\begin{pmatrix} 9 \\ -1.4 \end{pmatrix}$
  [Sidenote: there are more general "vector" concepts around (5xxx level?). Math 2: finite order!]

LA preview 1: Matrices and vectors – the "loose talk" of it

How to write them?

- **Notation/fonts.** Often distinguished from numbers. Possible ways (vectors in minuscles, matrices in capitals):

textbook:	$\mathbf{a}$, $\mathbf{A}$	"blackboard bold":	$\mathbb{b}$, $\mathbb{B}$
physics:	$\vec{c}$, $\vec{C}$	econ handwriting?	$\overline{d}$, $\overline{D}$
stats handwr:	$\underset{\sim}{x}$, $\underset{\sim}{X}$	maths: often	$y \in \mathbb{R}^n$, $Y \in \mathbb{R}^{m \times n}$

  - Suggestion: *overarrow* for vectors. What about matrices? (For speed, I suggest *you* use overbar. Add arrowheads later?)
- **Letters I will use:** "You must handle anything." However:
  - Book: uses a lot of a, b, c; $\mathbf{a}, \mathbf{b}, \mathbf{c}$; $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Clearly distinguished in print, not so clear in handwriting.
  - Typical linear equation system: $\mathbf{Ax} = \mathbf{b}$.
  - Suggestion: use $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for vectors often, but not exclusively: will also need: triplet $\mathbf{v} = (x, y, z)$; prices $\mathbf{p}$, quantities $\mathbf{q}$ ... Greek minuscles α, β, ... always numbers; (hopefully) s, t too.

LA preview 1: Matrices and vectors – the "loose talk" of it

To the topics! What we shall cover:

- **Basic algebra:** Scaling, addition, and a suitable product
  - ... whenever well-defined ...
  - ... take care to note what you are *not* allowed to do!

  (We will cover vectors first.)
- **More about square** $(n \times n)$ **matrices:** determinants; existence/nonexistence of inverse; computing inverse, if exists.
- Applications to **linear equation systems.**
  - Algorithm to solve completely
  - How to tell w/o solving whether there is precisely one solution?
  - And if so: How to single out one element of the solution without having to solve all?

(Later: LA *could be useful* for the remainder of the course – though you will be free to avoid LA *notation/language*.)

LA preview 1: Matrices and vectors: vectors first

There are two kinds of vectors: Row vectors / columns vectors.

- **Typical convention(s):**
  - As long as one only needs vectors – no (other) matrices – then vectors are rows. (Typographically convenient, fits on the line.)
  - In applications when matrices are used, then (more common than not), vectors are columns *unless specified otherwise.*
- This could be confusing. But the literature (often) works that way, and therefore you should be able to handle it.

Therefore, the lectures will follow those notational conventions:

- Today, vectors will be rows. An n-vector $\mathbf{x}$ is a row $(x_1, x_2, \ldots, x_n)$.
- Next week, vectors will default to being columns. $\mathbf{x}$ will be a column, and $\mathbf{Ax}$ will make sense.

(It *is* curriculum to be able to handle both rows and columns.)

LA preview 1: (row) vectors, basic operations

An n-vector $\mathbf{x}$ – a.k.a. *a vector* $\mathbf{x} \in \mathbb{R}^n$ – is an *ordered list* $\mathbf{x} = (x_1, \ldots, x_n)$ of numbers.

Equality is considered element-wise, and "ordered" means that $(1,\ 4)$ is not the same as $(4,\ 1)$.

Definition: The **zero (/null) vector** $\mathbf{0}$ of order n, is the vector $(0, \ldots, 0)$. (Use your favourite vector notation, e.g. $\vec{0}$.)

More definitions: **Scaling** and **vector addition** are defined element-wise:

- **Scaling:** A vector can be scaled by a number, element-wise. Example ($n = 3$): $-2\ (4,\ -7,\ 3.1) = (-8,\ 14,\ -6.2)$.
- **Addition:** Two vectors *of the same order* (i.e., same n) are added like e.g. $(2,\ 3,\ 4) + (-e,\ -2, 0) = (2 - e,\ 1,\ 4)$.

Subtraction? $\mathbf{x} - \mathbf{y}$ defined as $\mathbf{x} + (-1)\mathbf{y}$. And $(-1)\mathbf{y}$ denoted $-\mathbf{y}$.

LA preview 1: (row) vectors, basic operations

Scaling and addition "follow nice rules". Examples:

$\alpha\mathbf{0} = \mathbf{0}$ and $0\mathbf{x} = \mathbf{0}$ and $1\mathbf{x} = \mathbf{x}$
$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) = \mathbf{z} + (\mathbf{y} + \mathbf{x})$. Write: $\mathbf{x} + \mathbf{y} + \mathbf{z}$.
$(\alpha + \beta)(\mathbf{x} + \lambda\mathbf{y} + \mathbf{0}) = (\alpha\lambda + \beta\lambda)\mathbf{y} + \alpha\mathbf{x} + \beta\mathbf{x}$
Subtraction: $\mathbf{x} - \mathbf{y} = \mathbf{x} + (-1)\mathbf{y}$. Also $\mathbf{x} - (-\mathbf{y}) = \mathbf{x} + \mathbf{y}$.
Downscaling: $\frac{1}{\alpha}\mathbf{x}$ is OK for $\alpha \neq 0$.

All these rules reduce to the rules for numbers if we let $n = 1$.

I will routinely say things like "*never* divide by $\mathbf{x}$", totally dropping the qualification "... except $n = 1$ ...".

Graphical representation of scaling and addition: to follow handwritten/-drawn in a few minutes. First, a third operation which your supermarked uses every day:

LA preview 1: (row) vectors, basic operations

The dot product of $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots y_n)$ (both n-vectors, same n) is denoted $\mathbf{x} \cdot \mathbf{y}$ and defined as the *number*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

[Other terms: "scalar product" or "((unweighted) Euclidean) inner product".]

Note: the dot product does "unit pricing": $\mathbf{p} \cdot \mathbf{q} =$ price times quanitity for each good, added up to the grand total.

Examples with notes:

$(-1,\ 2,\ 3) \cdot (4,\ -5,\ 6) = -4 + (-10) + 18 = 4.$

$(0,\ 1) \cdot (1,\ 0) = 0 + 0$. So the product can be zero even when none of the factors are! (Contrast with numbers!)

$(1,\ -1) \cdot (2,\ 2) = 0$. Even without a single zero element!

LA preview 1: (row) vectors, basic operations

The dot product cont'd. "Nice" rules first:

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

$(\alpha\mathbf{x}) \cdot (\beta\mathbf{y}) = (\alpha\beta)(\mathbf{x} \cdot \mathbf{y})$, we drop the parentheses: $\alpha\beta\ \mathbf{x} \cdot \mathbf{y}$.

$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$

$\mathbf{x} \cdot \mathbf{0} = 0$.

$\mathbf{x} \cdot \mathbf{x} > 0$ except if $\mathbf{x} = \mathbf{0}$.

Then the caveats:

- As seen: $\mathbf{x} \cdot \mathbf{y} = 0$, does *not* imply that any vector is zero.
- *So never divide by a vector!* Even if $\mathbf{p} \neq \mathbf{0}$, you can not cancel $\mathbf{p}$ from the formula $\mathbf{p} \cdot \mathbf{x} = 0$, nor from $\mathbf{p} \cdot \mathbf{y} = \mathbf{p} \cdot \mathbf{z}$.
  - (Think: Enter an exchange economy market endowed with $\mathbf{y}$ and can choose $\mathbf{z}$ subject to budget ...)
- Do not mix-up scaling and dotting! $(\mathbf{x} \cdot \mathbf{y})\mathbf{z}$ is a scaling of $\mathbf{z}$, and is not the same as $\mathbf{x}(\mathbf{y} \cdot \mathbf{z})$. Try examples ...

LA preview 1: next up

- We define (equality and) scaling, addition and dotting when all vectors are column vectors: analogously.
- Later: matrices, and (equality and) scaling and addition for matrices (analogously); matrix products (slightly different).

But now:

- Graphical representation of vectors in $\mathbb{R}^2$ (and $\mathbb{R}^3$)
- A little bit of geometry: in particular, the *budget hyperplane.*
  - The angles we are interested in, are 0 and ± 90 and ± 180 degrees, as they have significant economic interpretations. But this is not going to be physics or trigonometry!
  - Note/terminology: the *norm* of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
    - Fact: $\mathbf{x} \cdot \mathbf{y} = c\, \|\mathbf{x}\|\, \|\mathbf{y}\|$ for some $c \in [-1, 1]$.
      Most interesting: $c = \pm 1$ iff $\mathbf{x}$ and $\mathbf{y}$ are proportional.
    - For $n \leqslant 3$, $\|\mathbf{x}\|$ equals the "physical length".