

## LA lectures 2 and 3 (Matrices): Handwritten notes & slides

Like for vectors, there are some slides, and some handwritten.

This file compiles notes largely from LA lectures 2 and 3:  
handwritten parts first (largely opposite to lectures)

- PDF file pages 2–9: handwritten.
  - Lecture 3 started midway page 5.
  - Page 9 was the end of lecture 3, but could just as well be omitted; see notes for lecture 4 instead.
- Page 10 ff. are the slideset. Repaginated to fit PDF page numbers.
  - Update October 17th with a couple of “priority clarifications”: boxes on green background slides 14 and 21

## Matrices

Def:: a matrix of order  $m \times n$  is a rectangular array of  $m$  rows and  $n$  columns of numbers.

Ex  $\vec{h} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$

Elements: the numbers, indexed by position, starting top-left.  
 $h_{21} = 2e$

Can specify by elements. See slides.

Defs: Transpose:  $\vec{A}'$ : flip rows/cols, see slides.

Null matrix:  $\vec{O}_{m \times n}$ ,  $m \times n$  with all elements 0  
(Zero matrix)

Identity matrix  $\vec{I}_n$   $n \times n$ , equal to  $\begin{pmatrix} 1 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & 1 \end{pmatrix}$

elements  $(i,i) = 1$ , all others  $= 0$ .

main diagonal of a square ( $n \times n$ ) matrix.

Equality; Scaling;  
Addition

Defined element-wise. See slides.

$A + B$  require  $A, B$  to have same order.

Rules: "nice". See slides.

Vectors revisited:

Row vector:  $1 \times n$  matrix

Column vector:  $n \times 1$  matrix

From now on: Vectors default to columns:

Unless otherwise specified, " $\vec{x}$ " means column vector.

Rows:  $\vec{x}'$ . Transpose of column vector.

Might write, to save space:  $\vec{x} = (1, 2, 3)'$

## Matrix multiplication.

The matrix product  $\vec{A} \vec{B}$  is defined iff  
 $\vec{A}$  is  $m \times n$ ,  $\vec{B}$  is  $n \times p$   
# of cols                      # of rows.

If so,  $\vec{C} = \vec{A} \vec{B}$  is  $m \times p$  with

$$c_{ij} = \vec{r}_i \cdot \vec{k}_j, \text{ where}$$

$$\vec{r}_i = \text{row number } i \text{ from } \vec{A}$$

$$\vec{k}_j = \text{column number } j \text{ from } \vec{B}$$

Examples: the "bottom-leftmost" element of

$$\vec{I}_2 \vec{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{element } \#(2,1) = (0,1) \cdot (1,0) = 0$$

$$\vec{I}_2 \vec{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$$

$$\text{element } \#(2,1) = (0,1) \cdot (2018, 2e) = 2e$$

$$\vec{H} \vec{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix} \begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix}$$

$$2e \cdot 2018 - 9 \cdot 1.4 + 5 \cdot 25 =$$

$$\vec{H} \vec{H} = \begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix} \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$$

Element (3,1)

$$2018 \cdot 25 + 10e$$

Dot products as matrix products: same order  
 $\vec{u} \cdot \vec{v} = \vec{u}' \vec{v}$  (if  $\vec{u}, \vec{v}$  column-vectors)

Strictly speaking, this is a minor cheat - a common one that you are allowed to!

The left-hand side is a number. The right-hand side is a one-by-one matrix, i.e. a box with a single number in it. It is common not to distinguish between such a matrix  $A = (a)$  and the number  $a$ .

They are even used for scaling, e.g.  $\mathbf{b}'\mathbf{M}\mathbf{b}$  scales  $\mathbf{v}$  by the number  $\mathbf{b}'\mathbf{M}\mathbf{b}$ .

This does not involve any matrix product between  $\mathbf{b}$  and  $\mathbf{v}$ .

[Lecture #3 started here]

Note: If  $\vec{u}, \vec{v}$  columns, same order

$\vec{u} \vec{v}'$  is also defined

$n \times 1$   $1 \times n$   
match  
 $n \times n$

Multiplication "how to" (by example)

$$\vec{A} \vec{B} \text{ where } \vec{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix}, \vec{B} = \begin{pmatrix} 4 & 1 & 2 & 0 \\ 4 & 2 & 0 & 0 \\ 4 & 3 & 1 & 0 \end{pmatrix}$$

• Top-left element: [top row] [leftmost column] =  $(1 \ 1 \ 1) \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = (1, 1, 1) \cdot (4, 4, 4) = 12.$

• 1<sup>st</sup> col. of  $\vec{A}\vec{B}$ :  $A \begin{bmatrix} 1^{\text{st}} \\ \text{col} \\ \text{of } B \end{bmatrix} = A \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ . Keep  $\begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ , move down the rows of  $\vec{A}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 4+4+4 \\ 0\cdot 4+1\cdot 4+0\cdot 4 \\ 0 \\ 4\cdot 4+(-5)\cdot 4+6\cdot 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ 0 \\ 20 \end{pmatrix}$$

• Same for each column of  $\vec{B}$ :  
Col # j of  $\vec{A}\vec{B} = \vec{A} \begin{bmatrix} \text{col} \\ \# j \\ \text{of } B \end{bmatrix}$

$$2^{\text{nd}} \text{ column: } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+2+3 \\ 0+2+0 \\ 0 \\ 4 \cdot 1 - 5 \cdot 2 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \\ 12 \end{pmatrix}$$

$$3^{\text{rd}} \text{ column: } \vec{A} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = [\text{exercise}] = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 14 \end{pmatrix}$$

$$4^{\text{th}} \text{ col: } \begin{matrix} \vec{A} & \vec{0} & = & \vec{0} \\ 4 \times 3 & 3 \times 1 & & 4 \times 1 \end{matrix}$$

$$\text{product} = \begin{pmatrix} 12 & 6 & 3 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 12 & 14 & 0 \end{pmatrix}$$

$$\text{Ex: } \vec{A} \begin{matrix} \text{same } \vec{A} \\ \begin{pmatrix} 4 & 1 & 20 \\ 4 & 2 & \textcircled{2} & 0 \\ 4 & 3 & 10 \end{pmatrix} \end{matrix} \text{ changing a single element of } \vec{B}$$

$$\text{Only } 3^{\text{rd}} \text{ column changes, into } \vec{A} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 4 \end{pmatrix}$$

## About rules for matrix multiplication

- \* Some rules are "as you would expect",

$$\text{like } \vec{A}(\vec{B} + \vec{C}) = \vec{A}\vec{B} + \vec{A}\vec{C}.$$

See slides.

- \* Beware what you are not allowed to.

$$\rightarrow \vec{A}\vec{B} \neq \vec{B}\vec{A}$$

except by coincidence  
(special cases)

$\rightarrow$  ... not even when both  $\vec{A}, \vec{B}$  are  $n \times n$ .

$\rightarrow$  Don't divide by matrices

$\rightarrow \vec{A}^2 = \vec{0}$  does not imply  $\vec{A} = \vec{0}$ .

$\rightarrow$  Squaring only possible for  $n \times n$ .

- \*  $(\vec{A}\vec{B})' = \vec{B}'\vec{A}'$  (if orders match.)

skill: if  $\vec{C}$  has an inverse  $\vec{C}^{-1}$ , then

$$\vec{A}\vec{C} = \vec{D}\vec{C} \Leftrightarrow \vec{A} = \vec{D}.$$



## Briefly on inverse matrices.

Def'n: Given a matrix  $\vec{A}$ .

If there exists a matrix  $\vec{B}$  s.t.

$$\vec{A}\vec{B} = \vec{B}\vec{A} = \vec{I}$$

then we call  $\vec{B}$  the inverse of  $\vec{A}$ , and denote it  $\vec{A}^{-1}$ .

Fact 1: Only square matrices can possibly have inverses.

Fact 2: If  $\vec{A}$  is  $n \times n$

$$\text{and } \vec{A}\vec{B} = \vec{I}_n$$

$$\text{or } \vec{B}\vec{A} = \vec{I}_n$$

then  $\vec{A}^{-1}$  exists and equals  $\vec{B}$ .

Example: Because  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 2a+c & 2b+d \end{pmatrix}$

$$= \vec{I}_2 \text{ if } a=d=1, b=0, c=-2: \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$(\text{You don't need to calculate } \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix})$$

## LA lecture 2: Matrices: what for?

**What?** a rectangular array of  $m$  rows and  $n$  columns of numbers.

**What is it good for?** (Putting numbers in boxes, huh? Apart from compact notation as bookkeeping tool?)

- We shall write the linear equation system

$$\begin{array}{rcl} a_{11}x_1 + \dots + a_{1n}x_n & = & b_1 \\ & \vdots & \vdots \\ & & \text{as } \mathbf{Ax} = \mathbf{b} \\ a_{m1}x_1 + \dots + a_{mn}x_n & = & b_m \end{array}$$

- Yesterday: manipulation tools for vectors (budgets, ...).

Now: manipulation tools for matrices, and then: use them to solve and/or characterize the solution/solvability of  $\mathbf{Ax} = \mathbf{b}$ .

(Other courses? If a random vector  $\mathbf{X}$  has covariance matrix  $\mathbf{W}$ , then the random variable  $Y = \mathbf{c} \cdot \mathbf{X}$  (with  $\mathbf{c}$  nonrandom) has variance  $\mathbf{c} \cdot (\mathbf{W}\mathbf{c})$  ...)

## LA lecture 2: Matrices: definition

**Definition:** a matrix of order  $m \times n$  (read: “ $m$  by  $n$ ”) is a rectangular array of  $m$  rows and  $n$  columns of numbers.

- **Example:** This example matrix is  $2 \times 3$ :

$$\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$$

- **Elements:** The numbers, indexed by (rownumber, column number), indexing counted from top-left.
- **Notation:** We write  $a_{ij}$  for the elements of  $\mathbf{A}$ . (“ $a_{i,j}$ ” if needed). Example: for  $\mathbf{H}$  above,  $h_{21} = 2e$ .
- **Specification by elements:** We can specify a matrix by specifying the elements individually.

Examples: write down the  $3 \times 2$  matrices  $\mathbf{U}$  and  $\mathbf{V}$  defined by  $u_{ij} = i - j$  and  $v_{ij} = (-1)^{i+j}$ .

## LA lecture 2: Matrices: equality, transpose

from previous page:

$$\mathbf{U} = \begin{pmatrix} 1-1 & 1-2 \\ 2-1 & 2-2 \\ 3-1 & 3-2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

**Equality:** element-wise. We have  $\mathbf{A} = \mathbf{B}$  iff  $a_{ij} = b_{ij}$ , all  $i, j$  (orders must be the same).

**Definition:** the *transpose*  $\mathbf{A}'$  of the  $m \times n$  matrix  $\mathbf{A} = (a_{ij})$  is  $n \times m$  with elements  $b_{ij} = a_{ji}$ . We call  $\mathbf{A}$  *symmetric* if  $\mathbf{A}' = \mathbf{A}$ .

[The prime symbol is not a derivative. No confusion as long as we keep linear algebra and analysis separated!]

- **Example:** if  $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$ , then  $\mathbf{H}' = \begin{pmatrix} 2018 & 2e \\ 9 & -1.4 \\ 25 & 5 \end{pmatrix}$
- **Exercise:** Explain why  $(\mathbf{A}')' = \mathbf{A}$ .

## LA lecture 2: Vectors as matrices. Rows & columns.

**Vectors recast:** A row vector is a  $1 \times n$  matrix. A column vector is an  $m \times 1$  matrix.

- The transpose of a row vector is a column vector.  
The transpose of a column vector is a row vector.

- **Vectors default to columns** from now on.

To specify a row vector, I will use a prime.

- **Example:** The first row of the example matrix  $\mathbf{H}$  is  $\mathbf{r}'_1 = (2018 \ 9 \ 25)$ . Here,  $\mathbf{r}_1$  is a column, namely  $\mathbf{r}_1 = \begin{pmatrix} 2018 \\ 9 \\ 25 \end{pmatrix}$ .
- To save space, you can specify a column  $\mathbf{x}$  as e.g.  $\mathbf{x} = (1 \ 2 \ 3)'$ . (Or, comma-separated.)

Noticed? We can specify a matrix by its rows or its columns.

- **Example:**  $\mathbf{H}$  is given by its rows  $\mathbf{r}'_1 = (2018 \ 9 \ 25)$  and  $\mathbf{r}'_2 = (2e \ -1.4 \ 5)$ . (*Enumeration matters!!*) Alternatively, by its columns  $\mathbf{c}_1 = \begin{pmatrix} 2018 \\ 2e \end{pmatrix}$ ,  $\mathbf{c}_2 = \begin{pmatrix} 9 \\ -1.4 \end{pmatrix}$  and  $\mathbf{c}_3 = \begin{pmatrix} 25 \\ 5 \end{pmatrix}$ .

## LA lecture 2: square matrices will turn out quite significant

### Definitions: (“square” and “identity” are musts!)

- A matrix  $\mathbf{S}$  is **square** if it is  $n \times n$ . The elements  $s_{ii}$  (i.e.,  $s_{ij}$  with  $i = j$ ) are called the *main diagonal* elements.
- A (necessarily square) matrix  $\mathbf{S}$  is called **symmetric** if  $\mathbf{S}' = \mathbf{S}$ .
- A (necessarily symmetric) matrix  $\mathbf{D}$  is **diagonal** if  $d_{ij} = 0$  whenever  $i \neq j$  (the “off-diagonal” elements are zero)
- The *identity matrix*  $\mathbf{I}_n$  of order  $n \times n$  is diagonal with elements = 1 on the main diagonal.

$$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}; \quad \begin{pmatrix} 14 & 0 & 0 \\ 0 & -e & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(asterisks indicate the main diagonal; a diagonal matrix;  $\mathbf{I}_3$ .)

We often say “order  $n$  identity” rather than “ $n \times n$ ”, and write  $\mathbf{I}$  without subscript if order is understood.

## LA lecture 2: Matrices: scaling and addition.

First:

**Definition:** The order  $m \times n$  **zero (/null) matrix**  $\mathbf{0}_{mn}$  (denoted simply  $\mathbf{0}$  if order is understood), has all elements equal to zero.

**Scaling** is defined element-wise:  $\mathbf{C} = \alpha\mathbf{B}$  has elements  $c_{ij} = \alpha b_{ij}$ .

**Addition** is defined element-wise, provided  $\mathbf{A}$  and  $\mathbf{B}$  of same order:  $\mathbf{C} = \mathbf{A} + \mathbf{B}$  has elements  $c_{ij} = a_{ij} + b_{ij}$ . And:  $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$ .

*“Nice rules” for scaling and addition:* if all matrices are of same order,

$$\alpha\mathbf{0} = \mathbf{0} \text{ and } 0\mathbf{A} = \mathbf{0} \text{ and } 1\mathbf{A} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{C} + (\mathbf{B} + \mathbf{A}). \text{ Write: } \mathbf{A} + \mathbf{B} + \mathbf{C}.$$

$$(\alpha + \beta)(\mathbf{A} + \lambda\mathbf{B} + \mathbf{0}) = (\alpha\lambda + \beta\lambda)\mathbf{B} + \alpha\mathbf{A} + \beta\mathbf{A}$$

$$\text{Subtraction: } \mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}. \text{ And } \mathbf{A} - (-\mathbf{B}) = \mathbf{A} + \mathbf{B}.$$

$$\text{Downscaling: } \frac{1}{\alpha}\mathbf{A} \text{ is OK for } \alpha \neq 0.$$

$$\text{and for transposition: } (\alpha\mathbf{A})' = \alpha\mathbf{A}' \text{ and } (\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'. \quad 15$$

## LA lecture 2: matrix multiplication I

The matrix product  $\mathbf{AB}$  is defined only iff

*the number of columns of (the left)  $\mathbf{A}$  equals  
the number of rows of (the right)  $\mathbf{B}$ .*

That is:  $\mathbf{A}$  is  $m \times n$  and  $\mathbf{B}$  is  $n \times p$ . (Note where the “ $n$ ” occurs!)

**Definition:** The product  $\mathbf{C} = \mathbf{AB}$  of an  $m \times n$  matrix  $\mathbf{A}$  and  $n \times p$  matrix  $\mathbf{B}$ , is  $m \times p$  with

$$c_{ij} = \mathbf{r}_i \cdot \mathbf{k}_j, \quad \text{where}$$

$\mathbf{r}_i$  is the  $i$ th row of  $\mathbf{A}$ , and  
 $\mathbf{k}_j$  is the  $k$ th column of  $\mathbf{B}$ .

**Examples** / not? Let  $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$  and  $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Which are *well-defined* of  $\mathbf{I}_2\mathbf{I}_2$ ,  $\mathbf{HI}_2$ ,  $\mathbf{I}_2\mathbf{H}$ ,  $\mathbf{HH}$ ,  $\mathbf{HH}'$ ,  $\mathbf{H}'\mathbf{H}$ ?



## LA lecture 2: matrix multiplication II

Cont'd: Let  $\mathbf{H} = \begin{pmatrix} 2018 & 9 & 25 \\ 2e & -1.4 & 5 \end{pmatrix}$  and  $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Of each of those which are well-defined among  $\mathbf{I}_2\mathbf{I}_2$ ,  $\mathbf{H}\mathbf{I}_2$ ,  $\mathbf{I}_2\mathbf{H}$ ,  $\mathbf{H}\mathbf{H}$ ,  $\mathbf{H}\mathbf{H}'$ ,  $\mathbf{H}'\mathbf{H}$ : calculate the “bottom-leftmost” element.

**Example** (small): Could  $\mathbf{AB}$  be a  $1 \times 1$  matrix? Hint: dot product?

**Example** (“big”?): Calculate  $\mathbf{AB}$  where  $\mathbf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2 & \dots & 2 \end{pmatrix}$  has 2018 columns and  $\mathbf{B}$  is  $2018 \times 3$  with all elements  $b_{ij}$  equal to 1.

## LA lectures 2&3: matrix multiplication “howto”

What is the third row of the following matrix product? What is the fourth column? Then, start calculating from top-left:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 & 0 \\ 4 & 2 & 0 & 0 \\ 4 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ 0 & 0 & 0 & 0 \\ ? & ? & ? & 0 \end{pmatrix}$$

How to calculate? For the first column of the product, you need the left matrix and the first column of the right:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -5 & 6 \end{pmatrix} \begin{pmatrix} 4 & \blacksquare \\ 4 & \blacksquare \\ 4 & \blacksquare \end{pmatrix} = \begin{pmatrix} 4 + 4 + 4 & \blacksquare \\ 0 + 4 + 0 & \blacksquare \\ 0 + 0 + 0 & \blacksquare \\ 16 - 20 + 24 & \blacksquare \end{pmatrix}$$

Then on to the second column. Though you know a few “0” elements.

## LA lectures 2&3: multiplication rules

**Rules:** Let  $\alpha$  and  $\beta$  be numbers. Suppose  $\mathbf{A}$  being  $m \times n$ , and suppose for each formula that  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  have orders such that sums and products *are well-defined*. Then:

$$\mathbf{0}_{k,m}\mathbf{A} = \mathbf{0}_{k,n} \text{ and } \mathbf{A}\mathbf{0}_{n,p} = \mathbf{0}_{m,p}. \quad (\text{Note: orders of "0"!})$$

$$\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n \quad (\text{Note: orders of "I"!})$$

$$(\alpha\mathbf{A})(\beta\mathbf{B}) = (\alpha\beta)\mathbf{A}\mathbf{B}, \quad \text{we drop the parentheses: } \alpha\beta \mathbf{A}\mathbf{B}.$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C} \quad \text{and} \quad (\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{C}$$

- Note:  $\mathbf{A}\mathbf{B} + \beta\mathbf{B} = (\mathbf{A} + \beta\mathbf{I}_n)\mathbf{B}$  ( $\mathbf{A}$  necessarily  $n \times n$ .)

Take care *not* to write the ill-defined " $\mathbf{A}$  plus  $\beta$ " times  $\mathbf{B}$ .)

$$\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}, \quad \text{we drop the parentheses: } \mathbf{A}\mathbf{B}\mathbf{C}.$$

$$(\mathbf{A}\mathbf{B})' = \mathbf{B}'\mathbf{A}', \text{ so also } (\mathbf{A}\mathbf{B}\mathbf{C})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$$

**Multiplication of squares:**  $\mathbf{A}\mathbf{A}$  exists iff  $\mathbf{A}$  is square. For square matrices, we write  $\mathbf{A}^k$  for the  $k$ -fold product  $\mathbf{A}\mathbf{A} \cdots \mathbf{A}$  ( $k \in \mathbb{N}$ ).

**Small exercise:** Explain why  $\mathbf{A}\mathbf{A}'$  always exists and is symmetric.

## LA lectures 2&3: matrix multiplication: INVALID operations

Take care not to apply bogus rules:

- Matrix multiplication is *not* performed element-wise, not even when  $\mathbf{A}$ ,  $\mathbf{B}$  both  $n \times n$ . (Exercise: what if both are diagonal?)
- Except “by coincidence”,  $\mathbf{AB} \neq \mathbf{BA}$ .
  - Even when both products are well-defined and of the same order – i.e., both  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times n$  – the products are usually unequal. (Calculate:  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  ... ?)
  - Exercise: for numbers we have  $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$  and formulae for squares of sums/differences – are they valid if  $\alpha$  and  $\beta$  are replaced by  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$ ?
- Do not divide by matrices! Leave  $\mathbf{AC} = \mathbf{DC}$  as-is ... for now.
  - Later: criteria for when that is indeed  $\iff \mathbf{A} = \mathbf{D}$ . But even then, you cannot slash  $\mathbf{C}$  off  $\mathbf{CA} = \mathbf{BC}$  nor from  $\mathbf{ACA} = \mathbf{BCB}$ .
  - (But  $1 \times 1$ s that are (non-zero!) numbers? ... ?)
- It is possible that  $\mathbf{A}^2 = \mathbf{0}$  even when all  $a_{ij} \neq 0$ . Example:  
 $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ . (But  $\mathbf{A}'\mathbf{A} \neq \mathbf{0}$  for  $\mathbf{A} \neq \mathbf{0}$ , cf. dot product.)

## Some terminology, linear transformations and eq. systems

This slide quickly brushed over in lecture 3 and is not crucial.

(One important item that was originally here, is also in lecture 4 slides.)

Terminology: multiplication “does not commute”; Fix  $\mathbf{C}$ . To get  $\mathbf{LCR}$ , we “left-multiply by  $\mathbf{L}$ ” and “right-multiply by  $\mathbf{R}$ ”.

(Alternative phrases: pre-multiply/post-multiply.)

Matrix multiplication can be thought of as linear transformation, and the *only* linear transformations (“functions”) from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , are by some matrix multiplication taking  $\mathbf{x}$  in and returning  $\mathbf{Ax}$ .

- The Math2-relevant consequence: The only possible linear equations for  $n$  unknowns  $\mathbf{x}$ , are of the form  $\mathbf{Ax} = \mathbf{b}$ .
- Lecture 4: an algorithm to solve.

(End of typesetting ...)