## Linear equation systems: $\vec{A}\,\vec{x} = \vec{B}$

### Method: Gaussian Elimination.

First: · if and when an equation says

$$0 = [\text{something nonzero}]:$$

Stop, declare "no solution".

· if and when an equation says

$$0 = 0$$

delete it.

[If nothing left: All $\vec{x}$ solve.]

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Operations to use

↳ on eq. system or on $(\vec{A} \,\vdots\, \vec{B})$ "augmented coefficient matrix"

* Interchange / switch equations — rows of $(\vec{A} \,\vdots\, \vec{B})$
* ~~Sc~~ Scale an eq. or a row by a nonzero number.
* add a scaling of one eq. or one row to another eq / row.

Algorithm / cookbook

Start ~~to~~ with top-left element:

① If necessary & possible:

1st nonzero in the row

Interchange to get a nonzero leading element

↳ if not possible: move one step right.

② Scale to leading 1.

③ Eliminate below leading 1: by adding a scaling of the row to the ones below

④ Move down-right. On the part starting with that element: Do ① – ③

⑤ ① – ④ repeated until "staircase": zeroes 1

- each row has a leading one
- ~~..~~ ... with only 0 below it.

Now you can "solve bottom-up".

⑥ Eliminate upwards    See example.

Example: (matrix notation)

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$(\vec{A} \,\vdots\, \vec{b}) = \left(\begin{array}{ccc:c} 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right) \begin{array}{l} \\ -1 \\ \leftarrow \end{array} \sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \\ \cdot(-1) \end{array}$$

eliminate  Step 6.

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \\ -3 \; -1 \end{array}$$

Can solve bottom-up:
$z=0$, $y=4$,
$x = 2-4 = -2$.

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & \emptyset & 0 \end{array}\right) \begin{array}{l} \\ -1 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc:c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

the solution.

ie the eq. system $\vec{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$

## Example Matrix inversion

For each $t \in \mathbb{R}$, find the inverse of

$$\vec{A} = \begin{pmatrix} t & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix}$$

if it exists.

$$\vec{A}\,\vec{X} = \vec{I}$$

$$\left(\begin{array}{ccc|ccc} t & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

Could first: $-2$ from

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ t & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \quad -t \quad -2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2-t & 3-t & 1 & -t & 0 \\ 0 & -1 & 2 & 0 & -2 & 1 \end{array}\right) \quad \cdot(-1) \text{ then}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & 2-t & 3-t & 1 & -t & 0 \end{array}\right) \begin{array}{c} \\ t-2 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & 0 & 7-3t & 1 & t-4 & 2-t \end{array}\right) \begin{array}{c} \\ \\ \cdot \frac{1}{7-3t} \quad \text{for } 7-3t \neq 0 \end{array}$$

If $7=3t$: last eq says $0=$ nonzero.

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{7-3t} & \frac{t-4}{7-3t} & \frac{2-t}{7-3t} \end{array}\right) \begin{array}{c} \\ \\ 2 \quad -1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{-1}{7-3t} & \frac{11-4t}{7-3t} & \frac{t-2}{7-3t} \\ 0 & 1 & 0 & \frac{2}{7-3t} & 2\,\frac{3-2t}{7-3t} & \frac{t-3}{7-3t} \\ 0 & 0 & 1 & \frac{1}{7-3t} & \frac{t-4}{7-3t} & \frac{2-t}{7-3t} \end{array}\right) \begin{array}{c} \\ -1 \\ \\ \end{array}$$

error in lecture

$$\sim \left( I \;\middle|\; \underbrace{\begin{matrix} \frac{-3}{7-3t} & \frac{5}{7-3t} & \frac{1}{7-3t} \\ \frac{2}{7-3t} & \frac{6-4t}{7-3t} & \frac{t-3}{7-3t} \\ \frac{1}{7-3t} & \frac{t-4}{7-3t} & \frac{2-t}{7-3t} \end{matrix}}_{} \right)$$

$$= \vec{A}^{-1} = \underline{\underline{\frac{1}{7-3t} \begin{pmatrix} -3 & 5 & 1 \\ 2 & 2(3-2t) & t-3 \\ 1 & t-4 & 2-t \end{pmatrix}}}$$

This slide fixed after lecture.