LA 2019 lectures #5–#7

We have:

- Matrices and vectors basics
- Linear equation systems
  - Theory, and Gaussian elimination
- A bit about the inverse:
  - Definition
  - If $\mathbf{A}$ has an inverse, call it $\mathbf{A}^{-1}$, then the equation system $\mathbf{AX} = \mathbf{B}$ has precisely one solution, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
  - We can *calculate* $\mathbf{A}^{-1}$ by eliminating $(\mathbf{A}\big|\mathbf{I})$ down to $(\mathbf{I}\big|\mathbf{M})$

Lecture #5:

- Determinants!
  - "Definition" [loosely – what you need to know]
  - Properties
  - Two ways to calculate: matrix operations and cofactors.
  - Formula for inverse.

Lecture #6: how to calculate inverses using the formula. Slide 25ff.

LA 2019 lect.#5: The determinant ... what and ... ?

What? A special function that takes a *square* ($n \times n$) matrix as input, and returns a number.

History: used for linear eq. systems before matrices! Nine Chapters of Mathematical Art, China, $\approx$ 200 BCE ...

Example: The determinant of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $\quad ad - bc$.
(What happens if $a = f''_{xx}$, $b = c = f''_{xy}$, $d = f''_{yy}$?)

Does that number mean anything? It has an area/volume/ hypervolume interpretation you do not need to worry about. For the curious: diagram next page.

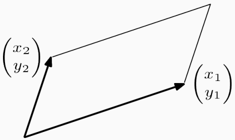
Fact, more important in this course. Let $\mathbf{A}$ be $n \times n$. Then:

$\mathbf{A}$ has an inverse *iff* its determinant is $\neq 0$

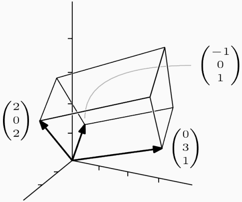
... and *iff* so, there is unique solution $\mathbf{x}$ to the system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

(Math 3 gives a version "on steroids", giving a very precise version of the counting rule.)

Geometry (OPTIONAL). If it helps your intuition – or curiosity: The determinant of $\mathbf{A}$ equals $\pm$ *the n-dimensional hypervolume of the n-dim. parallelogram spanned by the columns of $\mathbf{A}$.* Or by the rows of $\mathbf{A}$. Illustration, $n = 2$ and $n = 3$:



Determinant $= \pm$ the area, resp. the volume



Source: Wikibooks "Linear Algebra/Determinants as Size Functions", diagrams derived by Nicholas Longo from Jim Hefferon's open-source linear algebra book, license: CC-BY-SA-2.5.

A square matrix $\mathbf{A}$ has an inverse as long as that "hypervolume" does not collapse to zero:

$n = 2$ as long as the columns are not on the same line. $n = 3$: as long as the columns are not in the same plane. Etc.

LA 2019 lect.#5: The determinant. How to define.

Notation: $\mathtt{det}\, \mathbf{A}$ or $|\mathbf{A}|$ or "bar delimiters": $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for $\mathtt{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Beware, it is not an absolute value – and could be negative.

How to define? There are $n! = n(n-1)\cdots 2 \cdot 1$ ways to select n elements with precisely one from each row and column. For each selection, multiply the elements. Switch sign on half the products, and add up. The "sign switching" follows this rule:

- The selection of all the main diagonal elements: do not switch.
- Every time two rows are interchanged: switch sign.

Example: 2×2 and 3×3, the "NE/SW" diagonal. Resp.: $\begin{pmatrix} \cdot & \star \\ \star & \cdot \end{pmatrix}$ and $\begin{pmatrix} \cdot & \cdot & \star \\ \cdot & \star & \cdot \\ \star & \cdot & \cdot \end{pmatrix}$. Would be the main diagonal if the 1st and last row were interchanged. Precisely one interchange $\leftrightarrow$ precisely one sign switch. Put a negative in front.

LA 2019 lect.#5: The determinant. Definition rarely used.

You will likely not use the definition except a very few cases:

- 1×1: From the definition, the determinant is the element. (Again: Not any absolute value! Not even writing the bars here ...)
- 2×2: From the definition, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Maybe also for the following, although they will also follow from the rules to follow next slides:

- *Triangular* matrices: upper (resp. lower) triangular $\leftrightarrow$ all zeroes below (resp. above) the main diagonal: *Determinant = the product of the main diagonal elements.* (Why? Every other selection of precisely one element from each row/col., contains a zero.)
- Some might use it on 3×3?
  - Skipping: "Sarrus rule" (only valid for 3×3).

LA 2019 lect.#5: The Sarrus rule (skipped in lecture) I

The Sarrus rule for calculating 3×3 determinants.

CAVEAT: NOT VALID for larger!

Look at the picture:

$$\begin{array}{ccc:cc}
a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{33} & a_{31} & a_{32}
\end{array}$$

To the left, matrix elements. To the right, the first two columns repeated.

$$\text{Determinant} = \begin{array}{l} {\color{blue} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}} \\ -\Big[{\color{red} a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}} \Big] \end{array}$$

- The blue ones (first line) are the triplets connected with lines Northwest–Southeast.
- The red ones – which get subtracted, 2nd line – are the triplets connected with dashes Northeast–Southwest.

Example: $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 7 & 0 & -5 \end{vmatrix}$. Becomes $\begin{array}{ccc:cc} 1 & 2 & 4 & 1 & 2 \\ -1 & 8 & 3 & -1 & 8 \\ 7 & 0 & -5 & 7 & 0 \end{array}$

when we write the elements and then repeat the first two columns.

The ones left as-is: start top-left, go south-east:

$\begin{array}{ccc:cc} \boxed{1} & \boxed{2} & 4 & & \\ & \boxed{8} & \boxed{3} & -1 & \\ & & \boxed{-5} & \boxed{7} & 0 \end{array}$

cyan product: $1 \cdot 8 \cdot (-5) = -40$
green product: $2 \cdot 3 \cdot 7 = 42$
b/w product: $4 \cdot (-1) \cdot 0 = 0$

The ones to subtract/change sign are the south-*west* connections:

$\begin{array}{ccc:cc} & & \boxed{4} & 1 & \boxed{2} \\ & \boxed{8} & 3 & \boxed{-1} & \\ \boxed{7} & 0 & \boxed{-5} & & \end{array}$

magenta product: $4 \cdot 8 \cdot 7 = 224$
b/w product: $1 \cdot 3 \cdot 0 = 0$
yellow product: $2 \cdot (-1) \cdot (-5) = 10$

Determinant $= -40 + 42 + 0 - \left[224 + 10\right] = -232.$

The *cofactor* κ_{ij} of element (i, j) of $\mathbf{A}$, is formed by

- Deleting row i and column j from $\mathbf{A}$ (that is, the row and column of said element)
- Calculating the determinant of the rest $((n-1) \times (n-1))$
- Multiplying by $(-1)^{i+j}$.

Example: The cofactor κ_{23} of element $(2, 3)$ of $\begin{pmatrix} 3 & 9 & -2 \\ -1 & 5 & \boxed{7} \\ -11 & 4 & 7 \end{pmatrix}$ is (note how nothing magenta appears):
$(-1)^5 \begin{vmatrix} 3 & 9 \\ -11 & 4 \end{vmatrix} = -(12 - (-99)) = -111.$

The "$(-1)^{i+j}$": chessboard $\begin{pmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ \vdots & & & & \ddots & \end{pmatrix}$

Fact: For any $i = 1, \ldots n$, we have $|\mathbf{A}| = a_{i1}\kappa_{i1} + \ldots + a_{in}\kappa_{in}$.
IOW, we can calculate the determinant as follows:

- Pick one row number i (and stick to it!)
- For each element in the row, multiply it by its cofactor
- Add up.

This is called *cofactor expansion along the ith row.*

Example: $\begin{vmatrix} 3 & 9 & -2 \\ 2 & 5 & -1 \\ -11 & 4 & 7 \end{vmatrix}$ along the 2nd row: $2\kappa_{21} + 5\kappa_{22} + (-1)\kappa_{23}$

Need $\kappa_{21} = (-1)^{2+1} \begin{vmatrix} 9 & -2 \\ 4 & 7 \end{vmatrix} = -(63 - (-8)) = -71$

and $\kappa_{22} =$ [you go ahead!]: $(-1)^{2+2} \begin{vmatrix} 3 & -2 \\ -11 & 7 \end{vmatrix} = -1$

and $\kappa_{23} =$ [from previous slide] $= -111$

Answer: $2 \cdot (-71) + 5 \cdot (-1) - 1 \cdot (-111) = -36$.

Fact: $|\mathbf{A}'| = |\mathbf{A}|$. So we have $|\mathbf{A}| = a_{1j}\kappa_{1j} + \ldots + a_{nj}\kappa_{nj}$, for any j. That is *cofactor expansion along the* j*th column.*

Example: $\begin{vmatrix} 3 & 9 & -2 & 1 \\ 2 & 5 & -1 & 0 \\ -11 & 4 & 7 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix}$ along the 4th column.

$$1 \cdot (-1)^{1+4} \underbrace{\begin{vmatrix} 2 & 5 & -1 \\ -11 & 4 & 7 \\ 0 & 0 & -1 \end{vmatrix}}_{\text{expand along 3rd row}} + 0 + 0 + 1 \cdot (-1)^{4+4} \underbrace{\begin{vmatrix} 3 & 9 & -2 \\ 2 & 5 & -1 \\ -11 & 4 & 7 \end{vmatrix}}_{=-36,\text{ prev. slide}}$$

$$= (-1)^5 \cdot (-1) \begin{vmatrix} 2 & 5 \\ -11 & 4 \end{vmatrix} - 36 = 8 + 55 - 36 = 27$$

You can choose which row or which column. If there is one with a lot of zeroes, if often pays off to choose that one. (Although, ...)

For $n \times n$ matrices, the following rules apply:

(H1&2) Cofactor expansion applies along any row/column.

(H2) $|\mathbf{A}'| = |\mathbf{A}|$.

(H3) $|\mathbf{AB}| = |\mathbf{A}| : \cdot |\mathbf{B}|$. (Beware: both must be square, not just $\mathbf{AB}$.)

(H4) If one row/column is zero, the determinant is zero.
- Also if two rows / two columns are proportional, by (H6)

(H5) If you scale one row by t, then you scale the determinant by t. Same with "column".

(H6) The determinant does not change if a scaling α of a row is added to a *different row.* Same with "column".

(also H4) So if two rows are proportional, then $|\mathbf{A}| = 0$. Or if "two columns are proportional". But not "a row and a column".

(H7) Interchange two rows of $\mathbf{A}$, and you switch sign on the determinant (but keep the absolute value!). Same with "two columns". But do not try to interchange a row with a column!

Note: There is no simple rule for $|\mathbf{A} + \mathbf{B}|$.

Exercises: What is $|t\mathbf{I}_n|$? What is $|t\mathbf{A}|$?

Beware the common error! Answers: t^n resp. $t^n|\mathbf{A}|$.
Why? Scaling *just one* row by t will scale the determinant by t;
and, scaling the entire matrix $\leftrightarrow$ scaling every one of them.
For $n \times n$ matrices, there are n rows.

Recall that in Gaussian elimination, we had three operations on rows. On determinants, they do the following:

- Interchange two rows: switches sign.
- Scale a row: scales the determinant.
- Add a scaling of a row to another: no change.

Example: $\begin{vmatrix} 222 & 333 & 444 \\ 555 & 666 & 777 \\ 1111 & 2222 & 3333 \end{vmatrix} = 111 \cdot 111 \cdot 1111 \cdot \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix}$

and $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 0$

Exam: You can be asked, e.g. "Calculate this *without using cofactor expansion*."

LA 2019 lect.#5: The determinant. Elementary column op's

For determinants we can also do the same operations on columns.
But do not apply column operations to equation systems!

$$\text{Example: } \begin{array}{c} \begin{array}{ccc} \cdot\,{}^1\!/_{11} & \cdot\,{}^1\!/_{11} & \end{array} \\ \begin{vmatrix} 22 & 33 & 14 \\ 55 & 66 & 13 \\ 11 & 22 & 1 \end{vmatrix} \end{array} = 11^2 \begin{array}{c} \begin{array}{ccc} -1 & & + \\ -2 & + & \end{array} \\ \begin{vmatrix} 2 & 3 & 14 \\ 5 & 6 & 13 \\ 1 & 2 & 1 \end{vmatrix} \end{array} = 121 \begin{vmatrix} 2 & -1 & 12 \\ 5 & -4 & 8 \\ 1 & 0 & 0 \end{vmatrix}$$

which by cofactor expansion (along what, you think?) equals

$$= 121 \begin{vmatrix} -1 & 12 \\ -4 & 8 \end{vmatrix} = -121 \cdot 4 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$ (WHY?). Answer: 484.

(Worried you might be asked to calculate this w/o cofactor expansion? Not this one, for then you would in the end need to apply the definition on a 3×3, and ... but previous slide: an "easy" zero. Or we could ask that for something that ends up in $\beta \cdot |\alpha \mathbf{I}|$.)

LA 2019 lect.#5: The determinant. Debugging hint?

Observation *(not a typical exam question, but maybe good for "debugging your determinants"):*
Let precisely one element of $\mathbf{A}$ be "t". Then the determinant is of the form $\alpha t + \beta$.

- Generally, the determinant function is affine in each element: Cofactor expansion: $a_{ij}\kappa_{ij}+$ [no a_{ij} elsewhere].
  So for a, b_1 and t: each of these enters (and linearly!) in precisely one element. As a function of t: $\gamma t + \delta$, etc.
- The determinant is linear in each row and in each column.
  So c_1 enters (as a first-order term) *twice* but both are in the same row. Do cofactor expansion along that row, and you see that the determinant must be of the form $\eta c_1 + \epsilon$.

But a parameter that enters in several rows/columns, may have higher order. (Ex.: $|\lambda \mathbf{I}_n - \mathbf{A}|$ is an nth order polynomial in λ.)

Two applications in this course

[not stressed in class]

Fix a function $f \in C^2$ of n variables: $f(\mathbf{x})$. Let $\mathbf{H}$ be the so-called Hessian matrix: $h_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ (symmetric matrix, depends on $\mathbf{x}$)

- *Second-order approximation around $\mathbf{x}_*$:* the 2nd-order term will become $\frac{1}{2}(\mathbf{x} - \mathbf{x}_*)'\mathbf{H}_*(\mathbf{x} - \mathbf{x}_*)$ where $\mathbf{H}_*$ indicates that it is evaluated at $\mathbf{x}_*$. Prime denotes transpose; if you don't like that (with derivatives in the picture): $\frac{1}{2}(\mathbf{x} - \mathbf{x}_*) \cdot (\mathbf{H}_*(\mathbf{x} - \mathbf{x}_*))$. Note $\mathbf{H}_*()$ means product, not "of".

- Behind the scenes, this underlies the *2nd derivatives test* in n variables, and *concavity/convexity tests*. Case $n = 2$ in Math2:
  - The *Hessian determinant* $|\mathbf{H}|$ equals the "$AC - B^2$" from your first Math course's 2nd derivative test. (Math2 does not require you to use *matrix formulation* in your 2nd derivative tests, but the content is the same anyway!)
  - If both the $|\mathbf{H}|$ and the top-left element A are > 0:
    - everywhere, the function is convex
    - merely at some stationary point, then this is strict local min.
  - (Why do we have opposite signs $|\mathbf{H}| > 0 > A$ for concavity/max? Switch sign on f and thus on A; but since $n = 2$, then $|-\mathbf{H}| = (-1)^2|\mathbf{H}|$, no sign change! More than 2 variables: Math3!)

LA 2019: slides altered after lecture 5.

The draft slides had "extra" examples not covered in class.

- The first example now became the inverse example for lecture 6, and is moved.
- The next 4×4 example was never covered. Left in.

Recall expansion along the ith row:

- Pick one row number i (and stick to it!)
- For each element in the row, multiply it by its cofactor
  (...remember what a "cofactor" is, and in particular: the chessboard for signs)
- Add up.

Or pick a column instead of a row.

Supplementary: 4×4 cofactor expansion ex. from 2018. (I)

4×4 **example:** $\begin{vmatrix} 1 & 2 & 4 & -7 \\ -1 & 8 & 3 & 3 \\ 7 & 0 & -5 & 1 \\ 2 & 2 & -3 & 5 \end{vmatrix}$ in a boring way, no "clever" shortcuts.

First: pick one row or column. I pick column 4. Need the four cofactors (top to bottom, none involving column 4):

$$\kappa_{14} = (-1)^5 \begin{vmatrix} -1 & 8 & 3 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix} \qquad \kappa_{24} = (-1)^6 \begin{vmatrix} 1 & 2 & 4 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix}$$

$$\kappa_{34} = (-1)^7 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 2 & 2 & -3 \end{vmatrix} \qquad \kappa_{44} = (-1)^8 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 7 & 0 & -5 \end{vmatrix}$$

The determinant will be $-7 \cdot \kappa_{14} + 3 \cdot \kappa_{24} + 1 \cdot \kappa_{34} + 5 \cdot \kappa_{44}$.

But the cofactors involve 3×3 determinants that must be calculated. For example by cofactor expansion.

Supplementary: 4×4 cofactor expansion ex. from 2018. (II)

4×4 example cont'd: since this is a cofactor expansion exercise, use that method for every 3×3 too. Arbitrary choices: first row for the first two, second column for the last two.

κ_{14} by first row: $(-1)^5 \begin{vmatrix} -1 & 8 & 3 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix} =$

$(-1)^5 \cdot \Big\{ -1 \cdot \text{its cofactor} + 8 \cdot \text{its cofactor} + 3 \cdot \text{its cofactor} \Big\}$,
all cofactors relative to that 3×3 determinant (ignore for the moment that there was ever a 4×4).

$$(-1)^5 \cdot \Big\{ -1 \cdot \underbrace{(-1)^{1+1} \begin{vmatrix} 0 & -5 \\ 2 & -3 \end{vmatrix}}_{=0-(-10)} + 8 \cdot \underbrace{(-1)^{1+2} \begin{vmatrix} 7 & -5 \\ 2 & -3 \end{vmatrix}}_{=-(-21+10)} + 3 \underbrace{(-1)^{1+3} \begin{vmatrix} 7 & 0 \\ 2 & 2 \end{vmatrix}}_{=14-0} \Big\}$$

which equals $10 - 8 \cdot 11 - 3 \cdot 14 = -120$, and so the "$-7 \cdot \kappa_{14}$" contribution is $-7 \cdot (-120) = 840$. On to the three others.

Supplementary: 4×4 cofactor expansion ex. from 2018. (III)

4×4 example cont'd: $\kappa_{24} = (-1)^6 \begin{vmatrix} 1 & 2 & 4 \\ 7 & 0 & -5 \\ 2 & 2 & -3 \end{vmatrix}$ by first row.

$$1 \cdot (-1)^2 \left| \text{ [what?] } \right| + 2 \cdot (-1)^3 \left| \text{ [what?] } \right| + 4 \cdot (-1)^4 \left| \text{ [what?] } \right|$$

(Go ahead, fill in!)

$$1 \cdot (-1)^2 \underbrace{\begin{vmatrix} 0 & -5 \\ 2 & -3 \end{vmatrix}}_{=10} + 2 \cdot (-1)^3 \underbrace{\begin{vmatrix} 7 & -5 \\ 2 & -3 \end{vmatrix}}_{=-11} + 4 \cdot (-1)^4 \underbrace{\begin{vmatrix} 7 & 0 \\ 2 & 2 \end{vmatrix}}_{=14}$$

which sum up to $10 + 22 + 56 = 88$.

So the "$3 \cdot \kappa_{24}$" contribution is 264. Then what next?

Supplementary: 4×4 cofactor expansion ex. from 2018. (IV)

4×4 example cont'd: $\kappa_{34} = (-1)^7 \begin{vmatrix} 1 & 2 & 4 \\ -1 & 8 & 3 \\ 2 & 2 & -3 \end{vmatrix}$ by second column:

(The first "$-$" is the $(-1)^7$. The negative signs before the "2" are from the "chessboard of signs" for the 3×3):

$$-\left\{ -2 \begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} + 8 \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} \right\}$$
$$= -\left\{ -2(3-6) + 8(-3-8) - 2(3+4) \right\} = 96.$$

Contribution to determinant: $1 \cdot 96$. Then finally the contribution $5 \cdot \kappa_{44}$: κ_{44} is the same determinant as in the Sarrus example: -232. (Time ran out in class; finding it by expanding along the 2nd column $\rightsquigarrow$ exercise!) So the 4×4 determinant is $840 + 264 + 96 + 5 \cdot (-232) = 40$.

(And if you like Sarrus: yes, you can use it once you have reduced it to 3×3's.)

Back to the inverse: We have:

- Definition (and for square matrices, suffices to check ...)
- Exists iff $\mathbf{A}$ square AND $|\mathbf{A}| \neq 0$.
- How to find by Gaussian elimination

Next:

- A general formula based on cofactors
- Example(s) Lecture #5 instead covered Cramér briefly. Inversion by cofactors: examples in lecture #6.
- Rules! Old ones and some more.

A formula for the inverse: (book uses $\mathbf{C}$ and ...)

Fix $\mathbf{A}$ (square). Let $\mathbf{K}$ have elements κ_{ij}, where κ_{ij} is the cofactor of element (i, j) of $\mathbf{A}$. Then $\mathbf{K}'\mathbf{A} = \mathbf{A}\mathbf{K}' = |\mathbf{A}|\ \mathbf{I}$, so:

Provided $|\mathbf{A}| \neq 0$, $\boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{K}'}$ (beware: transpose!)

Terminology (not exam relevant): $\mathbf{K}' =: \mathtt{adj}(\mathbf{A})$, abbreviation for "adjugate". Or "adjunct" / "classical adjoint".

"Workload" for $\mathbf{K}'$:
n^2 cofactors, each cofactor is
$\pm$ an $(n-1) \times (n-1)$ determinant, each determinant is
a sum of $(n-1)!$ terms, ...

Beauty is in the eye of the beholder? (clickable – won't compute the inverse of this, but ...)

LA 2019 lect.#5: The inverse: about the formula

[not stressed in class] Just browsed through, and you won't be asked to reproduce this "proof" (which it really isn't, we have only postulated that cofactor expansion works). But see if you can *follow* the steps!

Again: If $|\mathbf{A}| \neq 0$, $\quad \boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathtt{adj}\,(\mathbf{A})} \quad$ where $\mathtt{adj}(\mathbf{A}) := \mathbf{K}'$

Why? The following is an exercise in determinant rules:

- Recall cofactor expansion by ith row: $|\mathbf{A}| = \sum_{\ell=1}^{n} a_{i\ell}\kappa_{i\ell}$
- If instead we picked the cofactors from another ("alien") row $h \neq i$, then we actually have $\sum_{\ell=1}^{n} a_{i\ell}\kappa_{h\ell} = 0$.
  Why? Since cofactors of row h do not depend on elements in row h, then this is the determinant of the matrix we get by replacing row h by a copy of row i. But that has two equal rows!
- To check $\mathbf{A}\mathbf{K}' = |\mathbf{A}|\,\mathbf{I}$, check each element (i, j); it equals row $\#i$ from $\mathbf{A}$ dot row $\#j$ from $\mathbf{K}$ (because the prime!) i.e. $\sum_{\ell=1}^{n} a_{i\ell}\kappa_{j\ell}$, which equals:

  $|\mathbf{A}|$ if $i = j$ (i.e. on the main diagonal), and 0 otherwise.
- Exercise: check element (i, j) of $\mathbf{K}'\mathbf{A}$. (Verifies that you only need to calculate "one of the products", if $\mathbf{A}$ is square.)

LA 2019 lect.#6: Calculate cofactors. 3×3 example (I)

Problem: Calculate the cofactors of $\mathbf{A} = \begin{pmatrix} 4 & 3 & \mathrm{p} \\ 1 & \mathrm{t} & \mathrm{q} \\ -2 & -3 & \mathrm{r} \end{pmatrix}$

This slide: two example cofactors:

The "p" element $(1, 3)$: strike out first row and third column, evaluate determinant of rest, (chessboard says: do not switch sign) $\begin{vmatrix} \text{[redacted]} & \text{[redacted]} & \text{[redacted]} \\ 1 & \mathrm{t} & \text{[redacted]} \\ -2 & -3 & \text{[redacted]} \end{vmatrix}$. That is, the cofactor is $-3 + 2\mathrm{t}$.

The "q" element $(2, 3)$: strike out second row and third column, "chessboard sign": $(-1)^{2+3} = -1$, cofactor $= - \begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix} = 6$.

Next: The full matrix $\mathbf{K}$. Blue elements: the ones calculated this slide. Red negative signs from the chessboard.

$\mathbf{A} = \begin{pmatrix} 4 & 3 & p \\ 1 & t & q \\ -2 & -3 & r \end{pmatrix}$, want the matrix $\mathbf{K}$ of cofactors:

$$\mathbf{K} = \begin{pmatrix} \begin{vmatrix} t & q \\ -3 & r \end{vmatrix} & -\begin{vmatrix} 1 & q \\ -2 & r \end{vmatrix} & 2t-3 \\ -\begin{vmatrix} 3 & p \\ -3 & r \end{vmatrix} & \begin{vmatrix} 4 & p \\ -2 & r \end{vmatrix} & 6 \\ \begin{vmatrix} 3 & p \\ t & q \end{vmatrix} & -\begin{vmatrix} t & q \\ -3 & r \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 1 & t \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} rt+3q & -r-2q & 2t-3 \\ -3r-3p & 4r+2p & 6 \\ 4q-pt & -rt-3p & 4t-3 \end{pmatrix}$$

For $\mathbf{A}^{-1}$, transpose into $\mathbf{K}'$ and scale by by $\frac{1}{|\mathbf{A}|}$: For $|\mathbf{A}|$:

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & p \\ 1 & t & q \\ -2 & -3 & r \end{pmatrix}, \text{ cofactors: } \mathbf{K} = \begin{pmatrix} rt+3q & -r-2q & 2t-3 \\ -3r-3p & 4r+2p & 6 \\ 4q-pt & -rt-3p & 4t-3 \end{pmatrix}$$

Then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathbf{K}'$. How to calculate $|\mathbf{A}|$?

By cofactor expansion,

- pick one row number i or one column number j;
- dot row number i of $\mathbf{A}$ with row i of $\mathbf{K}$

  or: dot column j of $\mathbf{A}$ with column j of $\mathbf{K}$.
- example: column #3.

  $|\mathbf{A}| = \left(\begin{smallmatrix} p \\ q \\ r \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 2t-3 \\ 6 \\ 4t-3 \end{smallmatrix}\right) = p(2t-3) + 6q + r(4t-3).$

Yields

$$\mathbf{A}^{-1} = \frac{1}{p(2t-3)+6q+r(4t-3)} \begin{pmatrix} rt+3q & -3r-3p & 4q-pt \\ -r-2q & 4r+2p & -rt-3p \\ 2t-3 & 6 & 4t-3 \end{pmatrix}$$

How does this formula give the expression for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$?

Clarification of wording: "Find an expression for" means, roughly: "valid as long as exists". This problem intended as example.

On to work: Remembering to transpose, cofactors are as follows:

of the "a" element: d — of the "c" element: $-b$

of the "b" element: $-c$ — of the "d" element: a

(the "$-$"s for the cofactors of elements $(2,1)$ and $(1,2)$: "chessboard".)

And so the answer is the familiar(?) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Back to wording: If asked for "an expression for the inverse", this is the answer – because it is an expression valid as long as the inverse exists. (Then, writing "as long as $ad \neq bc$" not required, but won't hurt.)

LA 2019 lect.#6: The inverse by cofactors: 4×4 example

Example (on board but different symbols): Use the formula to find *an expression for* $\begin{pmatrix} p_1 & 0 & 0 & p_2 \\ 0 & p_4 & p_3 & 0 \\ 0 & p_2 & p_1 & 0 \\ p_3 & 0 & 0 & p_4 \end{pmatrix}^{-1}$ (16 cofactors, each 3×3 – but we are in some luck ...)

Symmetries! Let $\varphi = \left|\begin{smallmatrix} p_1 & p_2 \\ p_3 & p_4 \end{smallmatrix}\right|$, the determinant of "the corners put together". Also happens to equal the "middle 2×2 block" (check sign!).

"The cofactors of the corners" have a common factor φ: We have $\kappa_{11} = p_4\varphi$ and $\kappa_{44} = p_1\varphi$. Beware signs: $\kappa_{14} = -p_3\varphi$ and $\kappa_{41} = -p_2\varphi$.

"The cofactors of the middle block" have a common factor φ: We have $\kappa_{22} = p_1\varphi$, $\kappa_{33} = p_4\varphi$ and (beware signs) $\kappa_{23} = -p_2\varphi$, $\kappa_{32} = -p_3\varphi$.

"The cofactors of the greens are all zero!" Each has two proportional rows or two proportional columns. Example: $\kappa_{12} = -\begin{vmatrix} 0 & p_4 & 0 \\ 0 & p_6 & 0 \\ p_7 & 0 & p_8 \end{vmatrix} = 0$. Remains:

- Put together the elements into $\mathbf{K}'$ (remember to transpose!)
- Calculate the determinant. We have all the cofactors.

LA 2019 lect.#6: The inverse by cofactors: examples (IV)

cont'd: The determinant is (say!) $p_1\kappa_{11} + 0 + 0 + p_3\kappa_{41}$

(Q: why no "$-$" before p_3?) – which $= p_1p_4\varphi - p_3p_2\varphi = \varphi^2$.

Stacking up cofactors, transposing:

$$\mathbf{K}' = \begin{pmatrix} p_4\varphi & 0 & 0 & -p_2\varphi \\ 0 & p_1\varphi & -p_3\varphi & 0 \\ 0 & -p_2\varphi & p_4\varphi & 0 \\ -p_3\varphi & 0 & 0 & p_1\varphi \end{pmatrix}$$

and the expression for the inverse is:

$$\frac{1}{\varphi}\begin{pmatrix} p_4 & 0 & 0 & -p_2 \\ 0 & p_1 & -p_3 & 0 \\ 0 & -p_2 & p_4 & 0 \\ -p_3 & 0 & 0 & p_1 \end{pmatrix}$$

Exercise: verify! (Multiply and see that you get the identity.)

(When multiplying to verify: do you spot why the corners $\leftrightsquigarrow$ inverse of the corners and the middle block $\leftrightsquigarrow$ inverse of the middle block?)

Terminology: "invertible" $\leftrightarrow$ has an inverse.

But also: A square matrix is called *non-singular* if it has an inverse, and *singular* if it does *not.*

("Singular" is fairly common. You can use "non-invertible" on the exam.)

Rules for the inverse. Known already:

(H8) If $\mathbf{A}$ is square and furthermore either $\mathbf{AM} = \mathbf{I}$ or $\mathbf{MA} = \mathbf{I}$, then $\mathbf{A}^{-1}$ exists and equals $\mathbf{M}$.

(H9) ... and if so: $\mathbf{M}^{-1}$ exists and equals $\mathbf{A}$. Consequently:

- If $\mathbf{A}$ is invertible, then so is $\mathbf{A}^{-1}$, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(H10) $\mathbf{A}$ is invertible $\Leftrightarrow$ square and $|\mathbf{A}| \neq 0$.

(H12) If so: formula $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathtt{adj}\,(\mathbf{A})$ where $\mathtt{adj}(\mathbf{A}) := \mathbf{K}'$

(H13) [part]... and the Gaussian elm'n method for $\mathbf{A}^{-1}$.

A few more rules next slide and we will be done with all formulae but eq. systems in (H13) and (H14) (Cramér's rule):

More rules for the inverse.

(also H10) Iff exists: $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$. (Because $1 = |\mathbf{I}| = |\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{A}|\ |\mathbf{A}^{-1}|$)

(also H8) $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$ provided any of them exists.

(also H9) Let $k \in \mathbb{N}$. Then $\mathbf{A}^k$ is invertible iff $\mathbf{A}$ is, and if so: $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$. Which we denote $\mathbf{A}^{-k}$.

- Easy to check. Determinant $\neq 0$ and $\mathbf{A} \ldots \mathbf{A}\mathbf{A}^{-1} \ldots \mathbf{A}^{-1} = \mathbf{I}$
- (Often seen notation, not exam relevant: $\mathbf{A}^0 = \mathbf{I}$, cf. $\alpha^0 = 1$.)

(H11) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$, then either $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ or neither side exists. Note reverse order!

- Similarly, $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$ or neither side exists. And also: $(c\mathbf{A})^{-1} = \mathbf{A}^{-1}(c\mathbf{I})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$ if $c \neq 0$.
- "Proof": Easy to check if well-defined: $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$.
- Beware: $(\mathbf{AB})^{-1}$ could exist even if $\mathbf{A}$ and $\mathbf{B}'$ are $m \times n$ with $m \neq n$.
  - Or more factors, e.g. $\mathbf{x}'\mathbf{A}\mathbf{x}$ is 1×1 and has an inverse iff $\neq 0$
  - (Not curriculum: That's why you in e.g. econometrics may see "large" products $(\ldots)^{-1}$ not written out. BTW, it requires $m < n$, impossible if $m > n$.)

LA 2019 lect.#6 auxiliary book problems examples

Example cases based on some book problems.

- Let $|\mathbf{X}'\mathbf{X}| \neq 0$. Show that $\mathbf{A} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ satisfies $\mathbf{A} = \mathbf{A}^2$. What can we say about $|\mathbf{A}|$?
  - $\mathbf{X}$ *not assumed square,* cannot simplify $\mathbf{A}$!
    But: $\mathbf{A}^2 = \mathbf{I} - 2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$
    which $= \mathbf{I} + (-2+1)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ because $\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{I}$.
  - $|\mathbf{A}| = |\mathbf{A}^2|$, so in Math 2 we can tell that $|\mathbf{A}|$ is zero or one.
    Actually zero! A proof that uses only Math2, but is too tricky for a Math2 exam: Suppose *for contradiction* that $\mathbf{A}^{-1}$ exists: Then $\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A}$, and $\mathbf{A} = \mathbf{I}$. And so $\mathbf{0} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Right-multiply by $\mathbf{X}$ to get: $\mathbf{0} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X} = \mathbf{X}\mathbf{I}$. But $\mathbf{X} = \mathbf{0} \Rightarrow |\mathbf{X}'\mathbf{X}| = 0$, contradiction.
- Suppose some power $\mathbf{B}^k$ is $\mathbf{0}$. Then $\mathbf{B}^{-1}$ does not exist, but $(\mathbf{I} - \mathbf{B})^{-1}$ does and equals $\mathbf{I} + \mathbf{B} + \ldots + \mathbf{B}^{k-1}$. (Cf. geometric series.)
  - $|\mathbf{B}^k| = |\mathbf{B}|^k$ and also $= |\mathbf{0}|$.
  - Calculate $(\mathbf{I} + \mathbf{B} + \ldots + \mathbf{B}^{k-1})(\mathbf{I} - \mathbf{B}) = [\ldots] = \mathbf{I} - \mathbf{B}^k = \mathbf{I}$.
- Suppose $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. Then $\mathbf{M}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$, all $k \in \mathbb{N}$.
  Show: Valid even for negative integers k iff $|\mathbf{D}| \neq 0$.
  Typically want: $\mathbf{D}$ diagonal. *Explain:* Why is this convenient?
  (Computes $\mathbf{M}^k$ when k large or ... sometimes even non-integer. Curriculum at BI Norwegian Business School and NTNU øk.ad. Application: $k = 1/2$ to standardize a random vector with covariance matrix $\mathbf{M}$.)

We have:

- Matrices and vectors basics
- Linear equation systems
  - Theory
  - How to solve
  - Theory
  - Calculations
- Inverses
  - Theory
  - Calculations
- Determinants

Remaining: back to eq. systems with square coefficient matrix

- Cramér's rule: Formula for a *single element* of the solution.
  - (Under the hood, no need to spend time on: just a different guise of the formula for the inverse.)
- Determinants and equation systems: a common problem type.

LA 2019 lecture #5 (briefly) and #7: unique solutions $\mathbf{A}^{-1}\mathbf{B}$, $\mathbf{A}^{-1}\mathbf{b}$, and Cramér's rule.

Assumed until nearly the end: $\mathbf{A}$ is square, $n \times n$.

- For this slide assume $\mathbf{A}$ invertible.
  - (H13) Then $\mathbf{AX} = \mathbf{B}$ has unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
  - ... gives you a formula in terms of $\mathbf{A}^{-1}$ (lengthy to calculate?)

Case: n eq's uniquely determining n variables as $\mathbf{A}^{-1}\mathbf{b}$

(H14) *Cramér's rule* gives a formula for *each individual element* of a unique solution $\mathbf{x}$: assume $\mathbf{b}$ a vector and $|\mathbf{A}| \neq 0$.

- Form the determinant D_j by replacing column j of $\mathbf{A}$ by $\mathbf{b}$.

  and calculating the resulting determinant, goes without saying?
- Then $x_j = D_j/|\mathbf{A}|$. (Well-defined, nonzero denominator)

Usage example: In a suitably big linear macro model [link to ECON2310], what if you are only interested in output Y? Or only consumption C?

Example: this system.

Q1: What is D_3?

Q2: What is $|\mathbf{A}_t|$?

Q3: If $t = 4$, what is z?

$$\underbrace{\begin{pmatrix} 4 & 3 & 2 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{pmatrix}}_{=\mathbf{A}_t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}.$$

Slide altered post lect.#5: For Q1&2, cofactor expand by col. 3, details slides 25–26. The cofactors: $\kappa_{13} = \begin{vmatrix} 1 & t \\ -2 & -3 \end{vmatrix} = -3 + 2t$;

$\kappa_{23} = -1 \begin{vmatrix} 4 & 3 \\ -2 & -3 \end{vmatrix} = 12 - 6 = 6$ and, $\kappa_{33} = \begin{vmatrix} 4 & 3 \\ 1 & t \end{vmatrix} = 4t - 3$

A1: Replace column 3 of $\mathbf{A}$ by the RHS $= (t, t, t)'$. Cofactor expand: $D_3 = t\kappa_{13} + t\kappa_{23} + t\kappa_{33} = t \cdot [-3 + 2t + 6 + 4t - 3] = \underline{\underline{6t^2}}$

A2: $2\kappa_{13} + (-1)\kappa_{23} + (-4)\kappa_{33} = 4t - 6 - 6 + 12 - 16t = \underline{\underline{-12t}}$

A3: When $t = 4$, $D_3 = 96$ and $|\mathbf{A}| = -48 \neq 0$, so $\underline{\underline{z = -2}}$.

Exercise: find the FLAWS in the arguments:

(I) "Let $\mathbf{A}$ be $n \times n$, and suppose that $\mathbf{b}$ is in fact one of the columns $\mathbf{A}$, namely number j. Then $D_j = |\mathbf{A}|$, and so x_j must be $= \frac{D_j}{|\mathbf{A}|} = 1$."

(II) "If the RHS were replaced by $(1, 1, 1)'$, then D_3 would be $6t$ and so $\frac{D_3}{|\mathbf{A}_t|} = -\frac{1}{2}$. The solution would always have $z = -\frac{1}{2}$."

The flaw: Cramér's rule requires $|\mathbf{A}| \neq 0$! If that were verified, the arguments would be valid. ((II): OK iff $t \neq 0$, so OK for $t = 4$.)

Exercise: What can you actually say about x_j in case (I)?
(Do you even know that a solution exists?)

Answer: there is a solution with $x_j = 1$ and all other $x_i = 0$ (check it!) – but if $|\mathbf{A}| = 0$, that solution is not unique; and, depending on $\mathbf{A}$, there *could* be solutions with $x_j \neq 1$ (dumb example: $\mathbf{0x} = \mathbf{0}$).

Back to the example from Cramér's rule.

Example problem: (has been frequent exam-type question)

Consider for each real constant t the matrix and the vector

$$\mathbf{A}_t = \begin{pmatrix} 4 & 3 & 2 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{pmatrix}, \qquad \mathbf{b}_t = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Questions:

(a) Calculate the determinant of $\mathbf{A}_t$, each $t \in \mathbb{R}$.

(b) For what $t \in \mathbb{R}$ will the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}_t$ have (i) no solution, (ii) precisely one solution, resp. (iii) several solutions?

Note: does not ask to "solve". (But the wording does not *forbid* solving either, so if that is all you know ...)

(a): We did that under the Cramér example slide 36: $\underline{\underline{-12t}}$.

(b): $\underline{\underline{\text{Unique solution iff } t \neq 0}}$. For $t = 0$: $\underline{\underline{\text{several solutions}}}$ (why? None or infinitely many, and since $\mathbf{b}_0 = \mathbf{0}$, there is at least one.)

Example variant 1:

What if the first row of $\mathbf{A}_t$ were replaced by $(4444,\ 4443,\ 4442)$? Recall that we can use elementary row operations for determinants, and note that both row 1 and row 3 "decrease by one as we move to the right"; the difference is $4446(1,\ 1,\ 1)$. Indeed, compute

$$\begin{array}{c}\begin{vmatrix} R-2 & R-3 & R-4 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix}\end{array}\begin{array}{l} \leftarrow\!\urcorner^{+} \\ \\ \llcorner_{-1} \end{array} = \begin{vmatrix} R & R & R \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix} = R \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & t & -1 \\ -2 & -3 & -4 \end{vmatrix}$$

For the $|\mathbf{A}_t|$ from the previous slide, we had $R = 6$ and got $-12t$. On this, we have $R = 4446$ and get a determinant equal to $\frac{4446}{6} \cdot (-12t) = -4446 \cdot 2t$ (or, just compute it! Exercise: use row operations as long as you can).

But this determinant too, is zero only iff $t = 0$ – and so part (b) would have the same answer for the variation on this page.

LA 2019 lect.#7: The determinant, and equation systems III

Example variant 2: with $(k\mathbf{A}_t)^k$ in place of $\mathbf{A}_t$. Here, $k \in \mathbb{N}$ a constant.

(a): $|(k\mathbf{A}_t)^k| = |k\mathbf{A}_t|^k = (k^3|\mathbf{A}_t|)^k = \underline{\underline{(-12tk^3)^k}}$. Did you remember the "3"?

(b): Same argument and same answer as the two previous slides!

Example variant 3: Replace $\mathbf{b}_t$ by $(1, 2, c)'$ (and, back to $k = 1$).
Q: How does that change anything, for each $c \in \mathbb{R}$?
A: (a) unchanged. (b): still $\underline{\underline{\text{unique solution for } t \neq 0}}$. For $t = 0$, Math2's only tool is starting to solve the inhomogeneous system:

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ -2 & -3 & -4 & c \end{array}\right) \begin{array}{l} \leftarrow + \\ -4 \quad 2 \\ \leftarrow + \end{array} \sim \left(\begin{array}{ccc|c} 0 & 3 & \boxed{6} & -7 \\ 1 & 0 & -1 & 2 \\ 0 & -3 & -\boxed{6} & c+4 \end{array}\right) \begin{array}{l} \leftarrow + \\ \\ \end{array}$$

Corrections after lecture: the LHS "6" and "−6".

First row then says $0 = c - 3$. $\underline{\underline{\text{Infinitely many solutions iff } t = 0 \ \& \ c = 3. \text{ If } t = 0 \neq c - 3\text{: no solution.}}}$ (Is it perfectly clear why?)

Example variant 4: What if you were also asked for *the degrees of freedom*?

Unique solution, hence no degrees of freedom, if $t \neq 0$ – this goes no matter what the RHS is! Therefore, for case $t = 0$, I change the example to general RHS $= (\alpha,\ \beta,\ \gamma)'$. The previous slide is a special case too. Doing the same operations:

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & \alpha \\ 1 & 0 & -1 & \beta \\ -2 & -3 & -4 & \gamma \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 3 & 6 & \alpha - 4\beta \\ 1 & 0 & -1 & \beta \\ 0 & -3 & -6 & \gamma + 2\beta \end{array}\right)$$

First row now says $0 = \alpha - 2\beta + \gamma$; if so, choose x_3 freely and see there is indeed solution. So: If $t = 0 = \alpha - 2\beta + \gamma$, solution with one degree of freedom. No solution if $t = 0 \neq \alpha - 2\beta + \gamma$.

Note: only case with the *maximum* n degrees of freedom, is $\mathbf{0x} = \mathbf{0}$. Here $n = 3$, so for $t = 0$: "One or two, at a glance." (Even: "one at a glance", because $n - 1$ degrees would imply all rows proportional ...)

Example variant 5: Let $\quad \mathbf{M} = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 0 & -1 \\ -2 & -3 & -4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$
($\mathbf{M}$ is $\mathbf{A}_0$ from the Cramér ex.)

(a) Calculate $\mathbf{Mv}$.

(b) Why does it follow from part (a) that $|\mathbf{M}| = 0$? You are *not* allowed to calculate the determinant by other means.

(a) [The "easy" part, cheap points. Matrix multiplication yields $\mathbf{0}$.]

(b) $\mathbf{Mv} = \mathbf{0}$ for some *nonzero* vector $\mathbf{v}$; hence the eq. system $\mathbf{Mx} = \mathbf{0}$ has more than one solution, and (as $\mathbf{M}$ square), $|\mathbf{M}| = 0$.

Note: If $\mathbf{w}$ solves the *homogeneous* system $\mathbf{Mx} = \mathbf{0}$ then $c\mathbf{w}$ also does, any $c \in \mathbb{R}$ – *but if* $\mathbf{w} = \mathbf{0}$*, this is no more than one vector!* For our *nonzero* $\mathbf{v}$, all the $c\mathbf{v}$ make infinitely many solutions.

Example (too tricky for exam?) yeah ... consider this optional.

Can determinants say anything about whether there is a solution to the eq. system (4 eq's, 3 unknowns, RHS=column) with

augmented coefficient matrix $\left(\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ -2 & -3 & -4 & 3 \\ 1 & 1 & 1 & c \end{array}\right)$?

Yes. Fact: if the *augmented* coefficient matrix has an inverse, then no solution exists. Note: does not say "iff"!

Why? An exercise in matrix manipulation:
Generally, $\mathbf{Ax} - \mathbf{b}$ can be written as $\mathbf{Mz}$, where $\mathbf{M} =$ the augmented coefficient matrix $(\mathbf{A}|\mathbf{b})$ ("written without the separator bar") and $\mathbf{z}' = (\mathbf{x}', -1)$ is the $n+1$-vector with $\mathbf{x}$ first and element $n+1$ being -1. To have $\mathbf{Ax} - \mathbf{b} = \mathbf{0}$, we want $\mathbf{Mz} = \mathbf{0}$, and if $|\mathbf{M}| \neq \mathbf{0}$, then this is only possible for $\mathbf{z} = \mathbf{0}$. But $z_{n+1} = -1$!

If $|\mathbf{M}| = 0$: eliminate on to contradiction or square coeff. matrix.

Not an exam question by itself, but potentially useful for the exam – especially the last "But", where lots of errors are made:

Can determinants say anything if fewer eq's than var's?

Yes. If there are m equations, coeff. matrix $= \mathbf{A}$, and we can find a nonzero $m \times m$ determinant from m columns from $\mathbf{A}$, then:

- choose the *other* variables free; "move them over to the RHS"
- now we have an $m \times m$ with nonzero determinant, in the m variables corresponding to the column numbers.
- (We knew this already for the situation when we have row-echelon form ("staircase") – but it holds more generally.)

Theory note: If there are fewer eq's than var's, then there cannot be unique solution!

But: There does not have to exist any solution at all!