# Implicit derivatives

Goal: If an eq. system

$$F(u, v, \vec{x}) = C$$
$$G(u, v, \vec{x}) = D$$

$C, D$ ← constants

determines $u, v$ as $C^1$ functions of $\vec{x}$: what are their derivatives

$$\frac{\partial u}{\partial x_i}, \quad \frac{\partial v}{\partial x_i} \quad ?$$

Tool: * Differentials

* Solving a linear eq. system 2×2 since the 2 $\begin{cases} F = C \\ G = D \end{cases}$

## This document:

* pages 1-5: lecture 2019-11-06 second hour.
* pages 6-12: lecture 2019-11-07. Page 30 shown in the end, not handwritten.
* pages 13-30: slides from 2018. Cookbook the same, some examples the same, additional examples given.

("Rules and formulas" exam attachment: page II)

Suppose $K$ changes by 0.1 and $M$ by $-0.3$

How much must $L$ change to maintain $q$? (output)

$dL$ = first-order approximation:

$$dL = \frac{-1}{f'_L}\left[ f'_K \, dK + f'_M \, dM \right]$$

$$= \frac{-1}{f'_L}\left[ 0.1 \, f'_K - 0.3 \, f'_M \right]$$

where all derivatives are evaluated at the $(K, L, M)$ where we are at.

Recall differentials: if $w = f(\vec{x})$,

$$dw = f'_{x_1}(\vec{x})\, dx_1 + \dots + f'_{x_n}(\vec{x})\, dx_n$$

- If $\vec{x}$ changes to $\vec{x} + d\vec{x} = \begin{pmatrix} x_1 + dx_1 \\ \vdots \\ x_n + dx_n \end{pmatrix}$

  then $dw$ is the first-order approximation of $\Delta w = f(\vec{x} + d\vec{x}) - f(\vec{x})$.

- "Advantages" over partial derivatives:
  → gathers all into one formula, which
  → allows for <u>simultaneous</u> changes.

Example: production $f(K, L, M) = q$.

Along an isoquant $dq = 0$: $dK, dL, dM$ related through

$$f'_K\, dK + f'_L\, dL + f'_M\, dM = 0$$

2 eq's example:

$$Y = C + I + G$$

$$C = f(Y).$$

Endogenous: $Y, C$

Q: If $G$ changes by (1),

≈ how much does $Y$ change?

Could: Insert, ← "lucky structure"

$$Y - f(Y) = I + G$$

$$(1 - f'(Y))dY = dI + dG$$

$$dY = \frac{dI + dG}{1 - f'(Y)}$$

$$\frac{\partial Y}{\partial G} = \frac{1}{1 - f'(Y)}$$

$$\Delta Y \approx \frac{1}{1 - f'(Y)} \cdot (1)$$

Can instead:

- Calculate differentials:

$$dY = dC + dI + dG$$

$$dC = f'(Y)\,dY$$

- Eq. system for $\begin{pmatrix} dY \\ dC \end{pmatrix}$:

2-vector $\begin{pmatrix} dY & - & dC \\ -f'(Y)dY & + & dC \end{pmatrix} = \begin{pmatrix} dI + dG \\ 0 \end{pmatrix}$ 2-vector

- Can solve out for $dY$, with/without matrix tools.

↑ add the eq's.

$$\begin{pmatrix} 1 & -1 \\ -f'(Y) & 1 \end{pmatrix} \begin{pmatrix} dY \\ dC \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dI \\ dG \end{pmatrix}$$

<u>Linear</u> in $\begin{pmatrix} dY \\ dC \end{pmatrix}$ !

## Problem to solve: Suppose

$$\begin{cases} F(u, v, x, y, z) = C \\ G(u, v, x, y, z) = D \end{cases}$$

← C, D const.

note: "arbitrary number of free variables" (x, y, z)

determines $u = u(x, y, z)$, $v = v(x, y, z)$ (as $C^1$ functions.)

two ⟺ two eq's.

* How to find the partial first derivatives of u and v?

## Cookbook

Notes:

- Calculate differentials
- Term-by-term or variable-by-variable, your preference.

① Differentiate the system

You get:

$$F'_u\,du + F'_v\,dv + F'_x\,dx + F'_y\,dy + F'_z\,dz = 0$$
$$G'_u\,du + G'_v\,dv + G'_x\,dx + G'_y\,dy + G'_z\,dz = 0$$

Everything evaluated at $(u, v, x, y, z)$

② Identify as linear eq system for du & dv.

② on matrix form:

$$\vec{A}\begin{pmatrix} du \\ dv \end{pmatrix} + \vec{B}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \vec{0}$$

where $\vec{A} = \begin{pmatrix} F'_u(u,v,x,y,z) & F'_v(u,v,x,y,z) \\ G'_u(u,v,x,y,z) & G'_v(u,v,x,y,z) \end{pmatrix}$

and $\vec{B} = \begin{pmatrix} F'_x( \quad ) & F'_y( \quad ) & F'_z( \quad ) \\ G'_x( \quad ) & G'_y( \quad ) & G'_z( \quad ) \end{pmatrix}$

③ <u>Solve</u> for $\begin{pmatrix} du \\ dv \end{pmatrix} = -\vec{A}^{-1}\vec{B}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

<u>Note</u>: take for granted that $\vec{A}^{-1}$ exists.

Matrix notation/formulation is optional; use whatever method you prefer.

④ <u>Read off</u> the partial derivatives.

Solving yields:

$$du = r_{11}\,dx + r_{12}\,dy + r_{13}\,dz$$
$$dv = r_{21}\,dx + r_{22}\,dy + r_{23}\,dz$$

this is $\frac{\partial u}{\partial y}$. Depends on $(u,v,x,y,z)$

Note: relation to 1 variable determined by 1 equation

$F(u, x, y, z) = C$

$$du = -\frac{1}{F'_u}\begin{pmatrix} F'_x & F'_y & F'_z \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$\frac{1}{F'_u}$ ⟷ $\vec{A}^{-1}$, $1\times 1$ here

$(F'_x \; F'_y \; F'_z)$ = $\vec{B}$, $1\times$ # of free variables

↑ # of dependent variables

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If the problem text were:

[eq. system] ... determines $u, v$ of $(x, y, z)$]

around the point P where $(u, v, x, y, z) = (1, 2, 3, 4, 5)$

(a) Differentiate the system ⟹ done

(b) [... find some derivatives ...]

→ (c) Approximate $v(\pi, 3.99, 5.05)$

$v(3,4,5) = 2$

$$v(\pi, 3.99, 5.05) \approx v(3,4,5) + (\pi - 3)\frac{\partial v}{\partial x}(3,4,5) + (-0.01)\frac{\partial v}{\partial y}(3,4,5) + 0.05\,\frac{\partial v}{\partial z}(3,4,5)$$

$\frac{\partial v}{\partial x}(3,4,5)$: have formula. Insert $u=1$, $v=2$, $x=3$, $y=4$, $z=5$.

Example:

Suppose the $C^1$ functions $u = u(x, y, z)$ and $v(x, y, z)$ satisfy

$$u^2 + v = xy + z$$
$$uv = y^2 - x^2$$

(a) Differentiate the system
(b) Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$

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Solution:
(a)

$$2u\,du + dv = y\,dx + x\,dy + dz$$
$$v\,du + u\,dv = 2y\,dy - 2x\,dx$$

(b) $u \cdot$[first] $-$ [second]: (eliminates $dv$)

$$(2u^2 - v)\,du = (uy + 2x)\,dx + (ux - 2y)\,dy + u\,dz$$

So $\frac{\partial u}{\partial x} = \frac{uy + 2x}{2u^2 - v}$, $\frac{\partial u}{\partial y} = \frac{ux - 2y}{2u^2 - v}$, $\frac{\partial u}{\partial z} = \frac{u}{2u^2 - v}$

Note: we did not solve for $dv$. (Because we didn't need

Possible shortcuts:

(c) Differentiate system → done.

| If now you are only asked for: | then you can |
| --- | --- |
| • derivatives of $u$ (not of $v$) | • eliminate $dv$, solve for $du$ only |
| • derivatives wrt. $y$ (not wrt anything else) | • put $dx = dz = 0$ before solving. But: do not zero out d[dependent variable] |
| • derivatives at a point say $(3, 9, -1, \pi, e)$ | • insert these numbers (after differentiating, before solving out) |

## Application:

Firm produces $Q(K, L)$,
chooses $K, L$ as stationary point of (maximum)

$$p\, Q(K, L) - cK - wL$$

Question: How will $K$ and $L$ change when $p$, $c$ and/or $w$ change?

– $(K, L)$ satisfy

$$p\, Q'_K = c$$

$$p\, Q'_L = w$$

"Simpler" if rewritten

$$Q'_K = c/p$$

$$Q'_L = w/p$$

Differentiate:

$$Q''_{KK}\, dK + Q''_{KL}\, dL = d\frac{c}{p} = \frac{1}{p}\, dc - \frac{c}{p^2}\, dp$$

$$Q''_{LK}\, dK + Q''_{LL}\, dL = \frac{1}{p}\, dw - \frac{w}{p^2}\, dp$$

$$\underbrace{\phantom{xx}}_{\vec{A}\begin{pmatrix} dK \\ dL \end{pmatrix}} \quad \text{where } \vec{A} = \begin{pmatrix} Q''_{KK} & Q''_{KL} \\ Q''_{LK} & Q''_{LL} \end{pmatrix}$$

$$\begin{pmatrix} dK \\ dL \end{pmatrix} = [\text{Hessian matrix}]^{-1} \, d\begin{pmatrix} c/p \\ w/p \end{pmatrix}$$

$$= \begin{pmatrix} Q''_{KK} & Q''_{KL} \\ Q''_{LK} & Q''_{LL} \end{pmatrix}^{-1} \begin{bmatrix} \frac{1}{p}\,dc - \frac{c}{p^2}\,dp \\ \frac{1}{p}\,dw - \frac{w}{p^2}\,dp \end{bmatrix}$$

$$= \frac{1}{Q''_{KK}Q''_{LL} - {Q''_{KL}}^2} \begin{pmatrix} Q''_{LL} & -Q''_{KL} \\ -Q''_{KL} & Q''_{KK} \end{pmatrix}$$

# Linearization, differentials, eq. systems, implicit derivatives ...

Recall the logic underlying the MR(T)S formula $\frac{\partial F/\partial K}{\partial F/\partial L}$:

(That's good enough "expression for" ... that clarification will be summarized at the end of tomorrow.)

- Fixing a *level curve* (isoquant, indifference curve, ...) $F(K, L) = C$, will determine one variable in terms of the other(s). Say, $L = L(K)$.
- *Total* derivative wrt. K: $\frac{\partial F}{\partial K} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial K} = 0$ (because C is constant, so $\frac{\partial C}{\partial K} = 0$). Solve out for $\frac{\partial L}{\partial K} = -\frac{\partial F}{\partial K}\Big/\frac{\partial F}{\partial L}$.

  (The MRS is the negative of this: how much must you increase one if you want to *reduce* the other.)

Topic for Tuesday (and likely a bit of Wednesday too):

- Suppose you have $n$ equations determining ("endogenizing"?) $n$ variables in terms of the other free (exogeneous, then?) variable(s). *What are the derivatives?*
  - Focus: $n = 2$. The "new stuff" arises as soon as $n > 1$.
  - Book: $(u, v)$ as functions of $(x, y)$ (or of $(x, y, z)$).
    But: you are expected to handle $(x, y)$ as functions of $(r, s, t)$ or $(K, L)$ as functions of $(p, w)$ or of $(p, w, a, b, \alpha, \beta, \gamma)$ or ...

# Linearization, differentials, eq. systems, implicit derivatives ...

If the $n$ equations were all *linear*, we would be able to solve (and know how!). But that's a very lucky case. Not so lucky example:

Production: sum of two Cobb–Douglases, $(1-\gamma)K^a L^b + \gamma K^\alpha L^\beta$.
Unit prices: $p$ and $w$ on $K$ and $L$, respectively.
$K$ and $L$ determined by first-order cond'ns for profit maximization:

$$(1-\gamma)aK^{a-1}L^b + \gamma\alpha K^{\alpha-1}L^\beta = p$$
$$(1-\gamma)bK^a L^{b-1} + \gamma\beta K^\alpha L^{\beta-1} = w$$

Q: How do $(K, L)$ depend on $(p, w)$? Or on everything else?

By Tuesday, we shall cover how to get *expressions for* the partial derivatives of $K$ and $L$.

(Did you suggest to solve the FOC's for $(K, L)$ explicitly, and then differentiate? Nah, only in very special cases you can. Since economics is so full of quantities *implicitly* given – for example in terms of FOC's – we need a way to handle derivatives of implicitly given functions.)

*Differentials.* ("Main tool" for avoiding too much new matrix-based terminology.)

- Input change from $(K, L)$ to $(K + \Delta K, L + \Delta L) \rightsquigarrow$ output change, *first-order approximated* to $F'_K(K, L)\,\Delta K + F'_L(K, L)\,\Delta L$
- The differential: If $Q = F(K, L)$ we *define* the differential $dQ$ of $Q$ as $\qquad F'_K(K, L)\,dK + F'_L(K, L)\,dL.$

  (Sometimes we just write $dF$ for $dQ$, identifying the "black box" with its output.)

  - The differential then obeys rules similar to the ones of derivatives: $d(Q + R) = dQ + dR$, $d(QR) = R\,dQ + Q\,dR$, and the chain rule: say, if $K$ and $L$ are functions of time $t$, then $dQ = F'_K(K, L)\,dK + F'_L(K, L)\,dL$ equals $F'_K(K, L)\,K'(t)\,dt + F'_L(K, L)\,L'(t)\,dt$.
  - The invariance property: the differential is "agnostic" as to whether a variable is free or dependent. The formula "$F'_K(K, L)\,dK + F'_L(K, L)\,dL$" remains valid if we "endogenize" $K$ and $L$; just insert the new formulae for $dK$ and $dL$.

# Linearization, differentials, eq. systems, implicit derivatives ...

The "differential form" has a couple of advantages:

- In the formula $F'_K(K, L)\,dK + F'_L(K, L)\,dL$, you can make *simultaneous changes* in $K$ and $L$.
- As the differential does not care what is "determined", then on the level curve (where $dQ = 0$) we can write the changes "without making a choice between $L = L(K)$ vs. $K = K(L)$": we have $F'_K(K, L)\,dK + F'_L(K, L)\,dL = 0$.
  - This says that – up to a first-order approximation accuracy – in order to stay at the level curve, the changes in $K$ and $L$ must be related that way. Which we can rewrite as $F'_L(K, L)\,dL = -F'_K(K, L)\,dK$ if we want to.
  - And if we at the end of a long night of model-building find out that we want to ask the question: "if I want to reduce $K$ by a small unit, how much must I then increase $L$ in order to fulfil my 100 pcs order?" – then the formula $dL = -\frac{F'_K(K,L)}{F'_L(K,L)}\,dK$ remains valid unless we divide by zero.

With more variables: $Q = F(K, L, M)$ (say): Differential now

$$F'_K(K, L, M)\,dK + F'_L(K, L, M)\,dL + F'_M(K, L, M)\,dM$$

If on a level curve AND $F'_L(K, L, M) \neq 0$, then

$$dL = \frac{-F'_K(K, L, M)}{F'_L(K, L, M)}dK + \frac{-F'_M(K, L, M)}{F'_L(K, L, M)}dM$$

We can change $K$ and $M$ *simultaneously* and this formula tells us $\approx$ how much $L$ must change in order to keep constant output. If you decide to consider $L$ as function of $K$ and $M$, then the *partial* derivatives are the respective coefficients:

$$\frac{\partial L}{\partial K} = \frac{-F'_K(K, L, M)}{F'_L(K, L, M)} \qquad \frac{\partial L}{\partial M} = \frac{-F'_M(K, L, M)}{F'_L(K, L, M)}$$

(and to get the respective MRS's: switch sign.)

So: writing with differentials, you can capture both partial and simultaneous changes.

Language/notation on this slide optional and voluntary; you can write the same content without vector notation on the exam.
(You have to know the same *content* in any case.)

Vector notation for differentials:

If $Q = F(\mathbf{x})$, then $dQ = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, dx_i$, which equals the dot product $\left(\frac{\partial F}{\partial x_1},\ \frac{\partial F}{\partial x_2},\ \dots,\ \frac{\partial F}{\partial x_n}\right) \cdot \left(dx_1,\ dx_2,\ \dots,\ dx_n\right)$ where it should really have been $\frac{\partial F}{\partial x_i}(\mathbf{x})$ (evaluation at $\mathbf{x}$) everywhere.

Typical notation: $\nabla F(\mathbf{x})$ for the *row* vector of partial first derivatives at $\mathbf{x}$, yields the matrix product form $\nabla F(\mathbf{x})\, d\mathbf{x}$.

This generalizes the univariate $F'(x)\, dx$. , and the upside-down triangle symbol saves us from using the prime symbol for neither transpose nor derivative ... the first-derivatives vector $\nabla F(\mathbf{x})$ is called the "gradient" of $F$ at $\mathbf{x}$.

Now, by saying that the language is optional and the content is not: you are indeed expected to be able to handle the $\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, dx_i$ – in fact, that is the multivariate chain rule, which you should know already before Math2 – so the only optional part is, we will not by any means require you to write it as "$\nabla F(\mathbf{x})\, d\mathbf{x}$".

# Linearization, differentials, eq. systems, implicit derivatives ...

**On to it:** Given constants $C$ and $D$ and $C^1$ functions $F$ and $G$. Consider the equation system

$$F(u, v, x, y, z) = C \qquad\qquad G(u, v, x, y, z) = D$$

Assume* that this equation system determines $u$ and $v$ as $C^1$ functions of $(x, y, z)$ if part (c) below is there, there will be some "around a point where"
[the equation system holds, say: where $(u, v, x, y, z) = (1, 2, 3, 4, 5)$]

Next up: a cookbook for their partial first derivatives.

In particular: for the following typical exam problem example:

(a) Differentiate the system†.

(b) Find a general expression for $\partial v/\partial y$.

(c) Approximate $v(3.1,\ 3.99,\ 5)$

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* That sentence spawns some questions: Does that "assume" indeed hold true, with $u$ and $v$ being $C^1$ (and defined!) everywhere? Not necessarily, but long story short and imprecise: as long as "all our calculations make sense", the functions will be *locally* defined and $C^1$ and our method will give the answer. Good enough for Math2!

† "Differentiate the system" means calculate *differentials*. Norwegian has two distinct words, "deriver" for derivatives vs. "differensier" for differentials; ~~thus, in order to convey the same information in both languages, an exam usually says e.g. "Differentiate the system (i.e., calculate differentials)" nowadays.~~ *obsolete info, exam English only (2019)*

**Cookbook for differentiating implicitly given functions,** $2 \times 2$ case: $F(u, v, x, y, z) = C, \qquad G(u, v, x, y, z) = D$

Want: the partial derivatives of the implicitly given $u$ and $v$. Cookbook essentially the same for any number of free variables. I chose 3 just because nothing says it must be the same number as eq's.

1. *Differentiate* the system. (Term by term or variable by variable, your choice.)

$$\begin{array}{llllll} F'_u\,du & +F'_v\,dv & +F'_x\,dx & +F'_y\,dy & +F'_z\,dz & =0 \\ G'_u\,du & +G'_v\,dv & +G'_x\,dx & +G'_y\,dy & +G'_z\,dz & =0 \end{array}$$

   Everything is evaluated at $(u, v, x, y, z)$, so these differentiated eq's could be a mess of $u$, $du$, $v$, $dv$, $x$, $dx$, $y$, $dy$, $z$, $dz$. Therefore:

2. *Identify* this as an equation system for $du$ and $dv$

$$\underbrace{\begin{pmatrix} F'_u & F'_v \\ G'_u & G'_v \end{pmatrix}}_{=:\mathbf{A}} \begin{pmatrix} du \\ dv \end{pmatrix} + \underbrace{\begin{pmatrix} F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{pmatrix}}_{=:\mathbf{B}} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   where $\mathbf{A}$ and $\mathbf{B}$ depend on $(u, v, x, y, z)$. (Nothing to “do” in this step, except catching what needs to be done – but if you do that, the rest is an algorithm.)

Matrix notation *not required* (but very useful on a slide):

3. *Solve* for $du$ and $dv$: $\begin{pmatrix} du \\ dv \end{pmatrix} = -\mathbf{A}^{-1}\mathbf{B}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ $= \mathbf{R}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

   and for once, this course does not ask you to check invertibility.

4. *Read off* the partial derivatives.

   Written out, we now have the form, where $(r_{ij}) = \mathbf{R} = -\mathbf{A}^{-1}\mathbf{B}$:

$$\begin{aligned} du &= r_{11}\,dx + r_{12}\,dy + r_{13}\,dz \\ dv &= r_{21}\,dx + r_{22}\,dy + r_{23}\,dz \end{aligned}$$

   and, e.g., $\frac{\partial u}{\partial z} = r_{13}$ and $\frac{\partial v}{\partial y} = r_{22}$. Note: all the $r_{ij}$ depend on $(u, v, x, y, z)$ (but you should *not* have any "$du$" left in your "$dv$" expression!)

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*Note: One equation, one determined variable redux:* If we have only $F = C$ and no "$v$", then we get $du = -\frac{1}{F'_u}\left(F'_x,\ F'_y,\ F'_z\right)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$. Recognize the analogues to the "$\mathbf{A}$" and "$\mathbf{B}$" matrices here!

Example problem had given a point $P$ with coordinates $(u, v, x, y, z) = (1, 2, 3, 4, 5)$ and a question:

(c) Approximate $v(3.1,\ 3.99,\ 5)$

From $(x, y, z) = (3, 4, 5)$ to $(x + dx, y + dy, z + dz) = (3.1,\ 3.99,\ 5)$ we find $dx = 0.1$, $dy = -0.01$, $dz = 0$. We have

$$\begin{aligned} v(3.1,\ 3.99,\ 5) &= v(3, 4, 5) + \Delta v \\ &\approx 2 + dv = 2 + 0.1 \cdot r_{21}\Big|_P - 0.01 \cdot r_{22}\Big|_P \end{aligned}$$

where the $\Big|_P$ indicates that you shall *insert for the coordinates:* $(u, v, x, y, z) = (1, 2, 3, 4, 5)$.

... question: why the “2” in “$\approx 2 + dv$”?

Simple example: Apply the cookbook to calculate $\partial C/\partial G$ when $(Y, C)$ are determined (as functions of $I$ and $G$) by:

$$Y = C + I + G, \qquad C = f(Y)$$

Here $f$ is some $C^1$ function with $0 < f' < 1$.

1. Differentiate: $dY = dC + dI + dG, \qquad dC = f'(Y)dY.$
2. OK, got it, we shall not solve for $Y$ ...
   You would probably not use matrices here? Exercise: do that, just to hone your LA skills.
3. $dC = f'(Y) \cdot (dC + dI + dG)$ yields $dC = \frac{f'(Y)}{1-f'(Y)}(dI + dG)$.
   And $dY = \frac{1}{1-f'(Y)}(dI + dG)$ if you want to follow the cookbook completely, but we do not need that for $\partial C/\partial G$. Exercise: instead of inverting the coefficient matrix, or using Gaussian elimination: what could you have used from the linear algebra curriculum to get out only $dC$?
4. The $dG$ coefficient in the solved-out expression for $dC$ is $\underline{\underline{\frac{f'(Y)}{1-f'(Y)}}}$.

The "for once, this course does not ask you to check invertibility" in step 3 $\rightsquigarrow$ OK to just divide in these problems, even if I hadn't written $f' < 1$.

**Example given in class:** Suppose the $C^1$ functions $u = u(x, y, z)$ and $v = v(x, y, z)$ satisfy $\quad u^2 + v = xy + z, \quad uv = y^2 - x^2.$

(a) Differentiate the system (i.e., calculate differentials)

(b) Find the three first-order partial derivatives of $u$.

(a) Calculate differentials: $\underline{\underline{2u\,du + dv = y\,dx + x\,dy + dz}}$ and $\underline{\underline{v\,du + u\,dv = -2x\,dx + 2y\,dy}}$

(b) Eliminate $dv$ from the differentiated system, e.g. by

$$\begin{aligned} 2u\,du + dv &= y\,dx + x\,dy + dz & \quad | \quad \cdot u, \text{ then } \leftarrow + \\ v\,du + u\,dv &= -2x\,dx + 2y\,dy & \quad -1 \end{aligned}$$

yields $(2u^2 - v)du = (uy + 2x)dx + (ux - 2y)dy + u\,dz$ and, in these particular problems you can divide w/o worrying over zeroness: $du = \underbrace{\dfrac{uy + 2x}{2u^2 - v}}_{= \partial u / \partial x}\,dx + \underbrace{\dfrac{ux - 2y}{2u^2 - v}}_{\partial u / \partial y}\,dy + \underbrace{\dfrac{u}{2u^2 - v}}_{\partial u / \partial z}\,dz$

more on the same example, and notes:

So we can just read off the derivatives as indicated:

$$\frac{\partial u}{\partial x} = \underline{\underline{\frac{uy + 2x}{2u^2 - v}}}, \qquad \frac{\partial u}{\partial y} = \underline{\underline{\frac{ux - 2y}{2u^2 - v}}}, \qquad \frac{\partial u}{\partial z} = \underline{\underline{\frac{u}{2u^2 - v}}}$$

- Though the cookbook would want you to solve for $du$ and $dv$, the question only asks for the first-order partial derivatives of $u$, and so $dv$ is not needed. (Just make sure you have eliminated it!)
- I asked a Q: if $z$ were not a variable, but a constant: would that affect $u'_x$?
  A: No; a partial change in $x$ is as if the other free variables – in this case $y$ and $z$ – were treated as constants.
  Note: This has nothing to do with $z$ not appearing in the expressions!

Then: *When and how can we speed up?* (Reading the problem helps!)
(And in time squeeze: get method right! Maybe not give *highest* priority to debugging expressions like slide 17...?)

- (Already done more than once:) If "part (b)" only asks for partial derivatives *of* one variable (say, $u$), then solving for $du$ gives you what you need. You can use Cramér if you like.
- If "part (b)" only asks for partial derivatives *with respect to* one free variable (say, $x$), then put $dy = dz = 0$
(the other *free* var's only! Do not delete the diff. of the *dependent* $dv$!)
  - Leave the answer to part (a) as-is, this is only for (b). In the example, when starting at (b) you can simplify to
  $2u\,du + dv = y\,dx \quad | \quad \cdot u$, then $\leftarrow +$
  $v\,du + u\,dv = -2x\,dx \quad -1$
  and so $(2u^2 - v)du = (uy + 2x)dx$ and $\frac{\partial u}{\partial x} = \frac{uy+2x}{2u^2-v}$ as we saw; then $\frac{\partial v}{\partial x} = y - 2u\frac{\partial u}{\partial x} = y - 2u\frac{uy+2x}{2u^2-v} = -\frac{yv+4xu}{2u^2-v}$
- If it asks for the derivatives *merely at a point,* then insert for point coord's at the beginning of "part (b)". But beware ...

cont'd next slide

## Linearization, differentials, eq. systems, implicit derivatives ...

- cont's: derivatives merely at point. Say, the example gives the point where $(u, v, x, y, z) = (1, 5, 2, 3, 0)$ and the question: "(b) Find $\frac{\partial u}{\partial x}(2, 3, 0)$ and $\frac{\partial v}{\partial x}(2, 3, 0)$."
  - Again, the answer to part (a) should be left as-is. and again, only derivatives wrt. $x$ are asked, so put $dy = dz = 0$. Furthermore, you can now insert point coordinates and work with the system $2\,du + dv = 3\,dx$ and $5\,du + dv = -4\,dx$. Subtracting, we have $-3\,du = 7\,dx$ and $\frac{\partial u}{\partial x}(2, 3, 0) = -7/3$; then $\frac{\partial v}{\partial x}(2, 3, 0) = 3 - 2\frac{\partial u}{\partial x}(2, 3, 0) = 3 + 14/3 = 23/3$.
- *Beware:* problem might say "around the point where $(u, v, x, y, z) = (1, 5, 2, 3, 0)$" and still ask for a general expression for $\frac{\partial u}{\partial y}$. Then the answer has "letters", ("$\frac{ux-2y}{2u^2-v}$")
  - Why give coordinates then? Consider one eq., $x^2 + y^2 = 2$. That does not define $y = y(x)$ (the circle is not a function graph). But "around the point where $(x, y) = (1, 1)$", it *does* define a function graph: the upper half-circle.

**New example** for Wednesday: The equation system

$$se^{y-x} + \ln(2t+y) + x = 3, \qquad ye^{-x} + stxy + t^2 = e^{-1}$$

defines continuously differentiable functions $x = x(s,t)$ and $y = y(s,t)$ around the point where $(s,t,x,y) = (2,0,1,1)$. (You shall not show this.)

(a) Differentiate the system (i.e., calculate differentials).

(b) Find a general expression for $\frac{\partial x}{\partial s}$.
Alternative question (b'): Calculate $\frac{\partial x}{\partial s}(2,0)$.

(a): $\underline{\underline{e^{y-x}\,ds + se^{y-x}(dy-dx) + \frac{1}{2t+y}(2\,dt+dy) + dx = 0}}$ and
$\underline{\underline{e^{-x}\,dy - ye^{-x}\,dx + txy\,ds + sxy\,dt + sty\,dx + stx\,dy + 2t\,dt = 0}}$.

(b): Only asked for $\partial x/\partial s$, so put $dt = 0$. Collect/reorder terms:
$(1-se^{y-x})\,dx + (se^{y-x} + \frac{1}{2t+y})\,dy = -e^{y-x}\,ds$ and
$(st - e^{-x})y\,dx + (stx + e^{-x})\,dy = -txy\,ds$. Now eliminate $dy$ – or use Cramér for $dx$:

new example cont'd: Cramér for $dx$ on the differentiated system

$$\begin{array}{lclcl} (1-se^{y-x})\,dx & + & (se^{y-x}+\frac{1}{2t+y})\,dy & = & -e^{y-x}\,ds \\ (st-e^{-x})y\,dx & + & (stx+e^{-x})\,dy & = & -txy\,ds \end{array} \qquad \text{(D)}$$

yields $dx = \dfrac{\begin{vmatrix} -e^{y-x}\,ds & se^{y-x}+\frac{1}{2t+y} \\ -txy\,ds & stx+e^{-x} \end{vmatrix}}{\begin{vmatrix} 1-se^{y-x} & se^{y-x}+\frac{1}{2t+y} \\ (st-e^{-x})y & stx+e^{-x} \end{vmatrix}} = \dfrac{\partial x}{\partial s}\,ds$ where

$$\frac{\partial x}{\partial s} = \underline{\underline{\frac{-(stx+e^{-x})e^{y-x}+txy\cdot\left(se^{y-x}+\frac{1}{2t+y}\right)}{(1-se^{y-x})(stx+e^{-x})-(se^{y-x}+\frac{1}{2t+y})\cdot(st-e^{-x})y}}}$$

(b'): If only asked for $\frac{\partial x}{\partial s}(2,0)$: Insert $(s,t,x,y)=(2,0,1,1)$ into (D), which then simplifies to $(1-2)dx+(2+1)dy=-ds$ and $-e^{-1}dx+e^{-1}dy=0$. From the latter, $dy=dx$ and so the former says $2dx=-ds$. Thus, $\frac{\partial x}{\partial s}(2,0)=\underline{\underline{-1/2}}$.

# Linearization, differentials, eq. systems, implicit derivatives ...

Finally, case $n > 2$ covered:

- Setup: $n$ equations[‡] $f_1(\mathbf{u}, \mathbf{x}) = C_1, \ldots, f_n(\mathbf{u}, \mathbf{x}) = C_n$ determining $\mathbf{u} \in \mathbb{R}^n$ as $n$ functions $u_1(\mathbf{x}), \ldots, u_n(\mathbf{x})$.
- Here, $\mathbf{x}$ are $m$ variables, $m$ could be any natural number.
- Cookbook: Differentiating this system gives the form

$$\mathbf{A}\,d\mathbf{u} + \mathbf{B}\,d\mathbf{x} = \mathbf{0} \quad \text{so that} \quad d\mathbf{u} = -\mathbf{A}^{-1}\mathbf{B}\,d\mathbf{x}$$

  where $\mathbf{A} = (a_{ij})_{i,j}$, $a_{ij} = \frac{\partial f_i}{\partial u_j}(\mathbf{u}, \mathbf{x})$ and $\mathbf{B} = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{u}, \mathbf{x})\right)_{i,j}$.

  (Just a bigger linear equation system. $n$ equations, $n$ unknowns $du_1, \ldots, du_n$.)
- Partial derivatives: $\partial u_i/\partial x_j$ = element $(i, j)$ of $\mathbf{R} := -\mathbf{A}^{-1}\mathbf{B}$.
- (Matrix notation: You might encounter in the literature – though certainly not on a Math2 exam – formulae like $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)^{-1} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$. Or with arrows: $\frac{\partial \vec{u}}{\partial \vec{x}} = -\left(\frac{\partial \vec{f}}{\partial \vec{u}}\right)^{-1} \frac{\partial \vec{f}}{\partial \vec{x}}$. )

  (And if you see the phrase "Jacobian" matrix, it is a matrix of *first-order* derivatives. Not the Hessian.)

[‡] Last-minute change from "F" to "f" due to the last bullet item: $\mathbf{f}$ is a column vector (not a matrix) of functions.