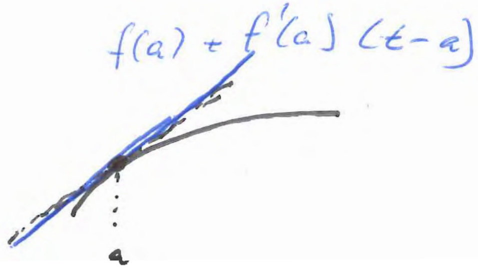
# First-, second- and higher order approximations

"linear" — "quadratic"

"known": $f(t) \approx f(a) + f'(a)\cdot(t-a)$, $t$ "near" $a$




$n$ var's $\vec{x}$: $f(\vec{x}) \approx f(\vec{a}) + f_1'(\vec{a})(x_1 - a_1)$

$+ \dots + f_n'(\vec{a})(x_n - a_n)$

vector notation: $f(\vec{a}) + \vec{\nabla} f(\vec{a})\,(\vec{x}-\vec{a})$

"gradient" $= (f_1'(\vec{a}), \dots, f_n'(\vec{a}))$

The RHS: matches function value at $t=a$ (at $\vec{x}=\vec{a}$)

matches first derivative at $t=a$

(all partial first-order derivatives at $\vec{x}=\vec{a}$)

Today: try to improve the approximation by a polynomial matching higher-order derivatives as well.

Fix function $f$.

The $k^{th}$ order approximation $P_{k,a}(t)$ of $f$ around $a$ is the $k^{th}$ order polynomial that matches all derivatives of order $0, \ldots, k$ at $a$.

order 0 = function value.

$n$ var's

$P_{k,\vec{a}}(\vec{x})$

around $\vec{a}$

all partial derivatives of order 0, ..., $k$

— o —

"Ex": Let $a = 0$, $k = 2$, $f(t) = e^t$.

$f(0) = 1$, $f'(t) = e^t$ so $f'(0) = 1$,

$f''(t) = e^t$ so $f''(0) = 1$.

$f(t) \approx 1 + t + \frac{1}{2}t^2 = P_{2,0}(t)$, write just $p(t)$

Check: $p(0) = 1$ OK

$p'(t) = 1 + 2 \cdot \frac{1}{2} \cdot t$ so $p'(0) = 1$, OK

$p''(t) = 1$ so $p''(0) = 1$, OK.

General $f(t)$, general $k$, $a=0$:

$$p(t) = P_{k,0}(t) = f(0) + f'(0)\,t + \tfrac{1}{2} f''(0)\,t^2 + \dots + \frac{1}{k!} f^{(k)}(0)\,t^k$$

where: $f^{(j)}$ is notation for $\left(\frac{d}{dt}\right)^j f$, the $j^{th}$ derivative

and $j! = j \cdot (j-1) \cdot \dots \cdot 1$.

Convention: $f^{(0)} = f$, $0! = 1$.

Check: does the $j^{th}$ derivative match?

$j=0$: OK: $f(0) + 0 + \dots + 0$.

$j>0$. Differentiate $j$ times:

$$0 + \underbrace{0 + \dots + 0}_{j-1} + \left(\frac{d}{dt}\right)^j \frac{1}{j!} f^{(j)}(0)\,t^j + \underbrace{\left(\frac{d}{dt}\right)^j \frac{1}{(j+1)!} f^{(j+1)}(0)\,t^{j+1} + \dots}_{\text{still have a } t^{\text{positive}} \text{ term.}}$$

$$f^{(j)}(0) \cdot \underbrace{\frac{j}{j!}}_{\frac{1}{(j-1)!}} \cdot \left(\frac{d}{dt}\right)^{j-1} t^{j-1} = \dots = \underbrace{f^{(j)}(0)}_{\text{OK}}.$$

Still have a $t^{\text{positive}}$ term.

$\to$ vanishes when $t=0$.

Ex: $e^t$ ($a=0$ still).

All $f^{(j)}(0)$ equal $1$.

$$P_k(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \dots + \frac{1}{k!}t^k.$$

Ex: $e^1$? Use $k=4$.

$$e^1 \simeq P_4(1) = 1 + 1 + \underbrace{\frac{1}{2} + \frac{1}{6} + \frac{1}{24}}_{\frac{12+4+1}{24}}$$

i.e. 2.7083333333...
The error is within 0.37 percent. Pretty good, compared to next page.

Ex: $e^{3t} - e^{2t}$, $a=0$, $k=3$

$f(0) = 0$, $f'(t) = 3e^{3t} - 2e^{2t}$, $f'(0) = 3-2 = 1$

$f''(t) = 9e^{3t} - 4e^{2t}$, $f''(0) = 9-4 = 5$

$f'''(t) = 27e^{3t} - 8e^{2t}$ $f'''(0) = 27-8 = 19$

$f^{(j)}(0) = 3^j - 2^j$

$$P_{3,0}(t) = 0 + 1 \cdot t + \frac{1}{2}5t^2 + \frac{1}{3!}19t^3$$

$$= t + \frac{5}{2}t^2 + \frac{19}{6}t^3$$

Ex: $\underbrace{\ln(1+t)}_{f(t),\; f(0)=0}$ around $t=0$. Approximate $\ln 2$, $k=4$?

at $t=0$:

$f'(t) = \frac{1}{1+t}$ $\quad 1$

$f''(t) = -\frac{1}{(1+t)^2}$ $\quad -1$

$f'''(t) = 2\frac{1}{(1+t)^3}$ $\quad 2$

$f^{(4)}(t) = -6\frac{1}{(1+t)^4}$ $\quad -6$

$$\ln(1+t) \simeq t + \frac{1}{2}(-1)t^2 + \frac{1}{3!}\cdot 2\cdot t^3 + \frac{1}{4!}(-6)t^4$$

$$= t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4$$

$$\ln 2 \simeq 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12-6+4-3}{12} = \frac{7}{12}$$

Comment:
This isn't a very good approximation; as k-> + infinity, the series p(1) - "the alternating harmonic series" - will converge slowly. How slow? It takes k=72 to get 1 percent accuracy: https://www.wolframalpha.com/input/?i=1-%281%2Fln+2%29*%28sum+%28-1%29%5E%28j%2B1%29%2Fj%2C+j%3D1+to+71%29 (clickable! You can change "72" to "721" for 0.1 percent if you are curious.)

If you click at https://www.wolframalpha.com/input/?i=maclaurin+series+ln%281%2Bt%29 you can see for t = 1 (and also to the right!) how the odd-numbered k=1 and k=3 polynomials overshoot and the even-numbered k=2 and k=4 undershoot the true value.

* These examples had $a = 0$. For $a \neq 0$, "move the coordinate system".

$$f(a) + f'(a)(t-a) + \frac{1}{2} f''(a)(t-a)^2$$

New function $g(s) = f(s+a)$.

$g(0) = f(a)$, $g'(0) = f'(a)$, ...

" $f'(a)(t-a) = g'(0)s$ ".

* More variables? $(\cancel{G2})/(G3)$ There is a matrix formulation for $k=2$:

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})\vec{z} + \frac{1}{2}\vec{z}\cdot(\vec{H}_a \vec{z})$$

$$\vec{z} = (\vec{x} - \vec{a}), \quad \vec{H}_{\vec{a}} = \begin{pmatrix} f''_{11}(\vec{a}) & \cdots & \cdots \\ \vdots & \ddots & \\ f''_{n1}(\vec{a}) & & f''_{nn}(\vec{a}) \end{pmatrix}$$

* For $k > 2$: Assume you want to approximate $f(\vec{b})$ from $f$ around $\vec{a}$. Let $g(t) = f(t\vec{b} + (1-t)\vec{a})$

$$f(\vec{b}) = g(1) \approx g(0) + g'(0)t + \dots + g^{(k)}(0)\frac{1}{k!}t^k.$$

$p$ is an approximation.

Q: → how good?

→ does it get better if we increase $k$?

→ Will $\lim_{k \to \infty} p_{k,a} = f(t)$?

A: "very often".

A: The error $f(t) - p_{k,a}(t)$ happens to be

$$= \frac{1}{(k+1)!} f^{(k+1)}(c)\,(t-a)^{k+1}, \quad \text{some } c \text{ between } a \text{ and } t.$$

Problem: We don't know $c$.

Ex: $e^t - p_{k,0}(t) = \frac{1}{(k+1)!} t^{k+1} \cdot e^c$ some $c$

So $e - p_k(1) = \frac{1}{(k+1)!} e^c$, some $c \in [0,1]$.

k=4 yields 1/120, guaranteeing an error less than 0.84 percent. True error: about half of that!

At worst: $e^c = e$. So ~~error~~ relative error $\left| \frac{e - p_k(1)}{e} \right| \le \frac{1}{(k+1)!}$

For more about this: Tuomas' lectures from last year, click here (from page 3) or the nect lecture, click here (first two pages)