ECON3120/4120 Mathematics 2

Wednesday January 11 2012, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

You are required to state reasons for all your answers. Throughout the problem set, you are permitted to use without proof information from a previous part, regardless of whether you managed to solve it or not.

Grades given run from A (best) to E for passes, and F for fail.

Problem 1 Weight: around 1/4.
In this problem, let t be a constant and consider the linear equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}$ in the unknown vector $\mathbf{x}$, where

$$\mathbf{A}_t = \begin{pmatrix} 2 & 3t-6 & 2 \\ t-4 & 12 & t-4 \\ 0 & t & 2t \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}.$$

(a) Show that $\mathbf{A}_t$ has determinant equal to $6t^2(6-t)$.

(b) For what values of the constant t will the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}$ have (i) unique solution, (ii) no solution, (iii) several solutions?

Problem 2 Weight: around 1/4.
The system of equations

$$a^x + a + by + e^{b-1} = 4$$
$$ab + \ln(ab^2) + e^{x+y-2} = 2$$

determines a and b as continuously differentiable functions of (x, y) around the point P : $(x, y, a, b) = (1, 1, 1, 1)$. (You are not supposed to show this).

(a) Differentiate the system.
(Hint: if you run into trouble here, it might be worth writing a^x as $e^{x \ln a}$.)

(b) Find a general expression for $\partial a/\partial y$.

(c) Find the partial elasticity of a with respect to x at the point P.

Problem 3 Weight: around 1/4.
Show that

$$\int (\ln t)^2 \, dt = t \cdot \big[(\ln t)^2 - 2\ln t + 2\big] + C,$$

and use this to find the general solution of the differential equation

$$\frac{\dot{x}(t)}{x(t)} = \left(\frac{\ln t}{\ln x(t)}\right)^2, \qquad t > 0,\ x > 1$$

Problem 4 Weight: around 1/4.
The following is a special case of a well-known problem of financial portfolio choice for a risk-averse investor:

$$\min_{x,y,z} \frac{x^2+y^2+z^2}{2} \qquad \text{subject to} \begin{cases} ax + by + cz &= d \qquad \text{(I)} \\ x + y + z &= w \qquad \text{(II)} \end{cases}$$

where a, b, c, d and w are constants, such that we do not have $a = b = c$ (the latter ensures that (I) and (II) do not contradict each other).

(a) State the Lagrange conditions associated to the problem. Is it clear whether a point which satisfies them, also will solve the (*minimization*) problem?

(b) Find the only possible solution of the problem.
Possible hint: You *may* (but are not obliged to) use without proof that the Lagrange conditions imply the following equation system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} + v \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{T}$$

where u and v are the Lagrange multipliers; then left-multiply (T) by the vector (a, b, c) and use (I) to obtain one equation for u, v, and then left-multiply (T) by $(1, 1, 1)$ to obtain another by applying (II). Then finally solve for u and v.

(Equation system (T) gives a version of the so-called «two-fund separation» property, which you may learn more about in ECON4510 or ECON5160.)