University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2012–12–11 exam

- This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading committee (updated with weighting information and, if applicable, bugfixes; any remaining errors are mine, not the committee's).
- Weighting: assigned at the grading committee's discretion. At this exam, the committee did apply equal weighting to each letter item, although with the interpretation that problem 2 should constitute two items (i.e. labeled 2 (a) and 2 (b)).
  (In case of appeals: the new grading committee assigns weighting at their discretion.)

Problem 1 Define for each real number t the matrix $\mathbf{A}_t$ by

$$\mathbf{A}_t = \begin{pmatrix} 1 & 1 & t^2 \\ 3t & 2t+18 & 9t \\ 3 & 2 & 0 \end{pmatrix}$$

and the matrix $\mathbf{B}_t = e^t \mathbf{A}_t$.

(a) The determinants of $\mathbf{A}_t$ and $\mathbf{B}_t$ are of the form $|\mathbf{A}_t| = a(t) \cdot (1-6t)$, resp. $|\mathbf{B}_t| = b(t) \cdot (1-6t)$. Find $a(t)$ and $b(t)$.

(b) For which value(s) of t does the equation system (where $\mathbf{x}$ is the unknown)

$$\mathbf{A}_t \, \mathbf{x} = \begin{pmatrix} 6t-1 \\ 12t-2 \\ 18t-3 \end{pmatrix}$$

have (i) unique solution, (ii) no solution, (iii) several solutions?

On the solution:

(a) For $|\mathbf{A}_t|$, cofactor expansion along the third column yields $t^2(6t-6t-54) - 9t(2-3) = 9t(1-6t)$ so that $\underline{\underline{a(t) = 9t}}$. Since $\mathbf{B}_t$ is 3×3, we have $|\mathbf{B}_t| = (e^t)^3 |\mathbf{A}_t|$, so $\underline{\underline{b(t) = 9te^{3t}}}$.

(b)
- Unique solution: if and only if $|\mathbf{A}_t| \neq 0$, i.e. if and only if $\underline{\underline{t \notin \{0, 1/6\}}}$.
- For $t = 1/6$, the right-hand side is null, so there is a solution and therefore several.
- For $t = 0$, the second equation yields $x_2 = -1/9$, while subtracting 3 of the first from the third, yields $-x_2 = 0$, a contradiction. No solution.

Problem 2 Evaluate the following integrals (hint: in both, you have to perform a substitution):

$$\text{(i)} \quad \int (x+1)^3 \, e^{x^2+2x+1} \, \mathrm{d}x \qquad\qquad \text{(ii)} \quad \int_{e^e}^{e^{e^e}} \frac{\mathrm{d}y}{y \cdot \ln y \cdot \ln(\ln y)}$$

On the solution:

(i) Substitute the exponent: $u = (x+1)^2$, with $\mathrm{d}u = 2(x+1)\mathrm{d}x$. The integral becomes

$$\frac{1}{2}\int ue^u \, \mathrm{d}u = \tfrac{1}{2}ue^u - \tfrac{1}{2}\int 1 \cdot e^u = \tfrac{1}{2}\, e^u(u-1) + C = \underline{\underline{\tfrac{1}{2}\, e^{x^2+2x+1}(x^2+2x) + C}}$$

(ii) The substitution $w = \ln \ln y$ (yielding $\mathrm{d}w = \mathrm{d}y/(y \ln y)$) will solve the integral; of course it is perfectly OK to first substitute $v = \ln y$ and then $w = \ln v$.

If one carries out the definite integral, then one *must* substitute limits of integration as well. Alternatively, one can calculate the indefinite integral and substitute back, as follows:

$$\int \frac{\mathrm{d}y}{y \cdot \ln y \cdot \ln(\ln y)} = \int \frac{\mathrm{d}w}{w} = \ln|w| + C = \ln|\ln \ln y| + C$$

yielding the answer

$$\ln|\ln\ln e^{e^e}| - \ln|\ln\ln e^e| = \ln|\ln e^e| - \ln|\ln e| = \ln e - \ln 1 = \underline{\underline{1}}$$

Problem 3 Define for $k > 0,\ \ell > 0$ the function

$$f(k,\ell) = \frac{1}{2}k^{1/2}\ell^{1/3} + \frac{1}{3}k^{1/3}\ell^{1/2} - (pk + q\ell)$$

Notice that the equation system (*) below, states the first-order condition for f to have stationary point at $(k,\ell) = (u,v)$.

(a) It is a fact (and you shall not prove) that the sum of concave functions, is concave. Show that f is concave.

On the solution of 3 (a): Consider $g(k,\ell) = Ck^a\ell^b$; then by some short calculations, $g''_{kk}(k,\ell) = g(k,\ell)\cdot a(a-1)/k^2$, $g''_{\ell\ell}(k,\ell) = g(k,\ell)\cdot b(b-1)/\ell^2$, and $g''_{k\ell}(k,\ell) = g''_{\ell k}(k,\ell) = g(k,\ell)\cdot ab/k\ell$, and the Hessian is $g^2\cdot\big[a(a-1)b(b-1)-a^2b^2\big] = abg^2\cdot[ab-a-b+1-ab] = ab(1-a-b)g^2$ which is positive when $ab>0$ and $a+b<1$, which is the case for both the two first terms of f, both having $a+b=5/6$. Furthermore, $g''_{kk} = g\cdot a(a-1)/k^2$ which is <0 if $C>0$. Therefore, each of those two terms are concave, and the sum is; f is then the sum of this concave sum and a linear (hence concave) and is therefore concave.

The equation system

$$\begin{aligned} \tfrac14 u^{-1/2}v^{1/3} + \tfrac19 u^{-2/3}v^{1/2} &= p \\ \tfrac16 u^{1/2}v^{-2/3} + \tfrac16 u^{1/3}v^{-1/2} &= q \end{aligned} \qquad (*)$$

defines u and v as continuously differentiable functions of p and q for $p>0$, $q>0$. (You are not supposed to prove this.)

(b) Differentiate the equation system (*) (i.e., calculate differentials).

(c) Calculate $\partial u/\partial p$ at the point where $p = 13/18$, $q = 2/3$ and $u = v = 1/64$.

(d) f has a maximum for $(k,\ell) = (u,v)$. Approximately how much does $f(u,v)$ change if p is increased by $\Delta p = 1/125$?

On the solution of 3 (b) ff.: (It is possible to recycle second derivatives from part (a), and since the mixed 2nd-derivatives are equal, the dv coefficient in the first equation will equal the du coefficient of the second. The following will not require this observation though.)

(b) Differentiating yields

$$\begin{aligned} -\big[\tfrac18 u^{-3/2}v^{1/3} + \tfrac{2}{27}u^{-5/3}v^{1/2}\big]\mathrm{d}u + \big[\tfrac{1}{12}u^{-1/2}v^{-2/3} + \tfrac{1}{18}u^{-2/3}v^{-1/2}\big]\mathrm{d}v &= \mathrm{d}p \\ \big[\tfrac{1}{12}u^{-1/2}v^{-2/3} + \tfrac{1}{18}u^{-2/3}v^{-1/2}\big]\mathrm{d}u - \big[\tfrac19 u^{1/2}v^{-5/3} + \tfrac{1}{12}u^{1/3}v^{-3/2}\big]\mathrm{d}v &= \mathrm{d}q \end{aligned} \qquad (**)$$

(c) Since we are only asked about a derivative at the point, then we can insert for the coordinates, that is, for $u = v = 1/64$. The candidates are certainly not expected to spot that the left-hand side is homogeneous of degree $-7/6$, but it is used below to simplify the calculations:

$$\begin{aligned} -\big[\tfrac18 + \tfrac{2}{27}\big]64^{7/6}\mathrm{d}u + \big[\overbrace{\tfrac{1}{12} + \tfrac{1}{18}}^{=5/36}\big]64^{7/6}\mathrm{d}v &= \mathrm{d}p \\ \big[\tfrac{1}{12} + \tfrac{1}{18}\big]64^{7/6}\mathrm{d}u - \big[\underbrace{\tfrac19 + \tfrac{1}{12}}_{=7/36}\big]64^{7/6}\mathrm{d}v &= \mathrm{d}q \end{aligned}$$

and we can eliminate $\mathrm{d}v$ by multiplying the first by 7, the second by 5 and adding up. Since $64^{7/6} = 128$:

$$128 \cdot \underbrace{\left(\tfrac{25}{36} - 7\left[\tfrac{1}{8} + \tfrac{2}{27}\right]\right)}_{=-151/6^3} \mathrm{d}u = 7\mathrm{d}p + 5\mathrm{d}q$$

so that at the point, $u'_p = -\frac{7\cdot 2^3\cdot 3^3}{128\cdot 151} = \underline{\underline{-\frac{27}{2416}}}$.

Alternatively, one may use Cramér's rule. The equation system is of the form $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathrm{d}u \\ \mathrm{d}v \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathrm{d}p \\ \mathrm{d}q \end{smallmatrix}\right)$ so that $\mathrm{d}u = \left|\begin{smallmatrix} \mathrm{d}p & b \\ \mathrm{d}q & c \end{smallmatrix}\right| /(ac - b^2) = (c\,\mathrm{d}p - b\,\mathrm{d}q)/(ac - b^2)$ – though we are only interested in a partial change in p, so we can put $\mathrm{d}q = 0$. We have $c = -7/36$, while the Hessian is $ac - b^2 = 128^2 \cdot \frac{151}{6\cdot 36\cdot 36} = 4 \cdot 151/3^5$ with the same conclusion.

Note: In either case, it should be noted that it is way more important to get the method right, than to avoid all the calculations mistakes.

(d) By the envelope theorem, the first-order approximation is

$$\Delta p \cdot \frac{\partial}{\partial p}\left[\frac{1}{2}k^{1/2}\ell^{1/3} + \frac{1}{3}k^{1/3}\ell^{1/2} - (pk + q\ell)\right]$$

evaluated at the point. We get

$$\frac{1}{125}(-k)\Big|_{k=1/64} = \underline{\underline{-\frac{1}{8000}}}$$

Problem 4 Consider the Lagrange problem (L), and the nonlinear programming problem (N):

$$\max\ e^{x-1} + e^{y-2} + e^{z-3} \quad \text{subject to} \quad 15x^2 + 12y^2 + 10z^2 = 900 \tag{L}$$

$$\max\ e^{x-1} + e^{y-2} + e^{z-3} \quad \text{subject to} \quad \begin{cases} 15x^2 + 12y^2 + 10z^2 = 900 \\ x \geq 0, \quad y \geq 0, \quad z \geq 0 \end{cases} \tag{N}$$

(a) State the Lagrange conditions associated with problem (L), and show that they are satisfied for the point $(x, y, z) = (4, 5, 6)$.

(b) Does $(x, y, z) = (4, 5, 6)$ satisfy the Kuhn–Tucker conditions associated to problem (N)?

(c) For each of the problems (L) and (N): Is it clear that a solution exists?

(d) You can assume without proof that $(x, y, z) = (4, 5, 6)$ solves problem (L). If problem (L) is modified by replacing 900 by 898, how much, approximately, does the optimal value change?

On the solution:

(a) For problem (L), the Lagrangian becomes

$$L = e^{x-1} + e^{y-2} + e^{z-3} - \lambda\big(15x^2 + 12y^2 + 10z^2 - 900\big)$$

and we get conditions as follows:

$$\begin{aligned} e^{x-1} &= \lambda \cdot 30x && (1)\\ e^{y-2} &= \lambda \cdot 24y && (2)\\ e^{z-3} &= \lambda \cdot 20z && (3)\\ 15x^2 + 12y^2 + 10z^2 &= 900 && (4) \end{aligned}$$

To show that they are satisfied at $(x, y, z) = (4, 5, 6)$, we first evaluate $15x^2+12y^2+10z^2 = 15 \cdot 16 + 12 \cdot 25 + 10 \cdot 36 = 240 + 400 + 360 = 900$, satisfying the constraint, and then verify that equations (1)–(3) yield the same λ:

$$\begin{aligned} &(1): && \lambda = e^{4-1}/120\\ &(2): && \lambda = e^{5-2}/120\\ &(3): && \lambda = e^{6-3}/120 \end{aligned}$$

We are done.

(b) For the Kuhn–Tucker problem, the Lagrangian becomes $K = L + \alpha x + \beta y + \gamma z$, but for the point $(x, y, z) = (4, 5, 6)$, the nonnegativity constraints are all inactive and $\alpha = \beta = \gamma = 0$. As the «$15x^2 + 12y^2 + 10z^2 \leq 900$» constraint is active, the Kuhn–Tucker conditions reduce to (i) stationarity of L (since $K = L$ when $xyz > 0$ – and L is stationary at $(4, 5, 6)$ by (a)) and (ii) $\lambda \geq 0$. We have already calculated $\lambda = e^3/120$, so the Kuhn–Tucker conditions hold.

(c) For each problem, the admissible set is (nonempty and) closed, and furthermore bounded (as the $15x^2 + 12y^2 + 10z^2 \leq 900$ is the inside of an ellipsoid), and the objective function is continuous. The extreme value theorem grants existence.

(d) The first-order approximation of the change is $(898 - 900) \cdot \lambda = -e^3/60$.