University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2018–06–08 exam

- New this semester: restricting calculators to the scientific calculator Casio FX-85EX (as well as a simpler arithmetic one).
- *Standard disclaimer:* This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- Weighting: at the committee's (and in case of appeals: the new grading committee's) discretion. The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice.
- Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

The committee (and in case of appeals, the new committee) is free to deviate.

Problems restated as given, followed by annotations boxed. The abbreviations «TP» (with problem number) refers to this semester's compulsory term paper problem set.

Problem 1 The equation system

$$\begin{aligned} xe^{x-sy} + ty + e^{-xy} &= 5 \\ se^{-x} + txy + e^{st} &= 1 \end{aligned}$$

defines continuously differentiable functions $x = x(s,t)$ and $y = y(s,t)$ around the point where $(s,t,x,y) = (0,2,0,2)$. (You shall not show this.)

(a) Differentiate the system (i.e., calculate differentials).

(b) Calculate $\dfrac{\partial y}{\partial s}(0,2)$.

On problem 1 Differentiating equation systems, and extracting derivatives from a differentiated system, is a recurrent problem-type considered straightforward. It was not covered in the term paper, being lectured later in the semester, but has been covered in three of the full exam sets assigned for seminars.

(a) Differentiating out yields

$$\begin{gathered} -xye^{x-sy}\,ds + y\,dt + \big((x+1)e^{x-sy} - ye^{-xy}\big)\,dx + \big(t - sxye^{x-sy} - xe^{-xy}\big)\,dy = 0 \\ \big(e^{-x} + te^{st}\big)\,ds + \big(xy + se^{st}\big)\,dt + \big(-se^{-x} + ty\big)\,dx + tx\,dy = 0. \end{gathered}$$

It is possible (but not at all required) to simplify the expressions. Also it is OK to differentiate term by term without collecting coefficients of $ds/dt/dx/dy$.

(b) Only a derivative at the point is asked for, so we can insert for $(s,t,x,y) = (0,2,0,2)$. Furthermore, only a partial derivative wrt. s is asked for, so $dt = 0$. Inserting, it simplifies to

$$\begin{gathered} 0\,ds + \big(1-2\big)\,dx + \big(2-0-0\big)\,dy = 0 \\ \big(1+2\big)\,ds + \big(-0+4\big)\,dx + 0\,dy = 0. \end{gathered}$$

that is: $2dy = dx$ and $4dx = -3ds$. Eliminating dx yields $dy = -\frac{3}{8}ds$, so the answer is $\underline{\underline{-3/8}}$.

Problem 2 Define for each real number t the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B}_t = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 0 \\ 4 & 0 & t \end{pmatrix}.$$

(a) Calculate $\mathbf{B}_t\mathbf{B}_t'$ and $\mathbf{B}_t(\mathbf{A} - s\mathbf{I})$, where s is a real constant, $\mathbf{I}$ is the 3×3 identity matrix, and the prime sign denotes matrix transpose.

(b)
- Use the following (there is *no* score for other methods!) to calculate $\mathbf{A}^{-1}$ or show that it does not exist: solve $\mathbf{AX} = \mathbf{I}$ by Gaussian elimination.
- Find $\mathbf{B}_t^{-1}$ or show that it does not exist, *without* using the method of the previous bullet item. (E.g., you can calculate cofactors.)

On problem 2 Compared to the term paper problem set: Matrix multiplication is considered a basic problem (TP3(a)) where the grave errors are elementwise multiplication and assuming commutativity. (And: it is intentional that those who do not know what «identity matrix» or «transpose» mean, must spend time to look it up in the book). Inverting by Gaussian elimination as mandatory method was given in TP 3(b). Here they are also asked to use a different method.

(a) $$\mathbf{B}_t\mathbf{B}_t' = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 0 \\ 4 & 0 & t \end{pmatrix}\begin{pmatrix} 0 & 1 & 4 \\ 0 & 2 & 0 \\ 3 & 0 & t \end{pmatrix} = \begin{pmatrix} 3^2 & 0 & 3t \\ 0 & 1^2+2^2 & 4\cdot 1 \\ 3t & 4\cdot 1 & 4^2+t^2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 9 & 0 & 3t \\ 0 & 5 & 4 \\ 3t & 4 & 16+t^2 \end{pmatrix}}}.$$

$$\mathbf{B}_t(\mathbf{A} - s\mathbf{I}) = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 0 \\ 4 & 0 & t \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -1 & 0 \end{pmatrix} - s\mathbf{B}_t = \begin{pmatrix} -6 & -3 & 0 \\ 1+8 & 2+10 & 3+12 \\ 4-2t & 8-t & 12 \end{pmatrix} - s\mathbf{B}_t$$

$$= \underline{\underline{\begin{pmatrix} -6 & -3 & -3s \\ 9-s & 12-2s & 15 \\ 4-2t-4s & 8-t & 12-st \end{pmatrix}}}$$

(b)
- Elementary row operations on the augmented coefficient matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -4 \quad 2 \\ \leftarrow + \\ \leftarrow\quad + \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 3 & 6 & 2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \rceil \\ \leftarrow + \end{array}$$

and then the last row becomes the impossible $(0 \;\; 0 \;\; 0 \;\; | \;\; -2 \;\; 1 \;\; 1)$. No solution, and thus $\underline{\underline{\mathbf{A}^{-1} \text{ does not exist}}}$.

- By cofactor expansion, $|\mathbf{B}_t| = 3\left|\begin{smallmatrix} 1 & 2 \\ 4 & 0 \end{smallmatrix}\right| = -24 \neq 0$, so $\mathbf{B}_t$ has an inverse for all t. Using cofactors, we get

$$\mathbf{B}_t^{-1} = \frac{-1}{24}\begin{pmatrix} 2t & -(t-0) & 0-8 \\ 0 & 0-12 & 0 \\ 0-6 & -(0-3) & 0 \end{pmatrix}' = \underline{\underline{\frac{-1}{24}\begin{pmatrix} 2t & 0 & -6 \\ -t & -12 & 3 \\ -8 & 0 & 0 \end{pmatrix}}}$$

Problem 3

(a)
- Calculate $\int q\, e^{9q}\, dq$
- Calculate $\int_r^{\infty} s^{-9} \ln s\, ds$ (where $r > 0$).

(b) Find the general solution of the differential equation $\dot{x}(t) + \dfrac{x(t)}{t} = \dfrac{3}{t}$.

On problem 3 Integration and differential equations have appeared in most exams in this course, a fact that has been communicated in seminar problem assignments.

(a)
- Use integration by parts with $u = q$, $v' = e^{9q}$. (It is hardly required to make explicit the $p = 9q$ substitution.) We get $q \cdot \frac{1}{9}e^{9q} - \int 1 \cdot \frac{1}{9}e^{9q}\, dq = \underline{\underline{C + \big(\frac{q}{9} - \frac{1}{81}\big)e^{9q}}}$.
- The integral is by definition equal to $\lim_{R\to+\infty} \int_r^R s^{-9} \ln s\, ds$. With now $u' = s^{-9}$, $v = \ln s$, we get

$$\lim_{R\to+\infty} \left(\left[\frac{s^{-8} \cdot \ln s}{-8}\right]_r^R - \int_r^R \frac{1}{-8} s^{-8} \cdot \frac{1}{s}\, ds \right) = \lim_{R\to+\infty} \left[\frac{s^{-8} \ln s}{-8} - \frac{s^{-8}}{(-8)^2}\right]_r^R$$

$\lim_{R\to+\infty} R^{-8} = 0$. We need $\lim_{R\to+\infty} \frac{\ln R}{R^8}$, which is an «$\frac{\infty}{\infty}$» form. l'Hôpital's rule yields $\lim_{R\to+\infty} \frac{R^{-1}}{8R^7} = 0$, and the answer becomes $\underline{\underline{\big(\frac{\ln r}{8} + \frac{1}{64}\big)r^{-8}}}$.

(b) This differential equation can be solved both as linear (as it stands) as well as separable (as $\dot{x} = \frac{3-x}{t}$). Either method is of course perfectly fine. The following chooses to treat it as separable: then we have a constant solution $x \equiv 3$, and otherwise $\frac{dx}{3-x} = \frac{dt}{t}$, which we can integrate as $-\ln|3-x| = K + \ln|t|$, which yields $3 - x = \pm e^{-K}/|t|$. The general solution is therefore $\underline{\underline{x(t) = 3 - D/t}}$. Here D is an arbitrary constant; $D = 0$ corresponds to the constant solution, and $D \neq 0$ to $\pm e^{-K}$; the manipulations of $\pm$\{absolute values\} is arguably not the most critical part of the answer.

Problem 4 Define for $x > 0$ and all real y the functions f and h by

$$f(x,y) = (y - e^{y-x}) \cdot \frac{\ln(1+x^2)}{x} \qquad \text{and} \qquad h(x) = f'_x(x,x) = \frac{2(x-1)}{1+x^2} + \frac{\ln(1+x^2)}{x^2}$$

(a)
- Show that $\lim_{x \to 0^+} h(x) = -1$.
- Find $\lim_{x \to +\infty} h(x)$.

(b)
- Show that $h(w) = 0$ for at least one $w \in (0,1)$. (You are not asked to compute w.)
- Let $w \in (0,1)$ be such that $h(w) = 0$ as in the previous bullet item. Take for granted that $f''_{xx}(w,w) > 0$.
  Show that $(x,y) = (w,w)$ is a saddle point for f.

(c) Consider the maximization problem

$$\max\ f(x,y) \qquad \text{subject to} \quad y \geq x, \quad x \geq 1 \tag{P}$$

- State the Kuhn–Tucker conditions associated with the problem (P).
- Show that if the Kuhn–Tucker conditions are satisfied at (x,y), then $x = y$.

Let now $a > 0$ be a constant and consider the function $g(x) = (a - e^{a-x}) \cdot \frac{\ln(1+x^2)}{x}$ for $x > 0$. (That is, $g(x) = f(x,a)$, with $a > 0$ taken as constant.) Take for granted that g has a global minimum point x_*.

(d) The minimum value $V = g(x_*)$ depends on a. Find an expression for $V'(a)$.

On problem 4 This problem is not unlike parts of Problem 1 (and 4, for (c)) of the December 2017 exam. Like that problem, significant parts correspond to questions in TP problems 1 and 2 – though, herein with functions that should be easier to handle than in the TP, given the information. It is intentional that errors in the more complicated derivative f'_x should not cause much trouble other than getting a wrong expression in the Kuhn–Tucker conditions: the function $f'_x(x,x) = h(x)$ is given as a formula; in (b), the sign of $f''_{xx}(w,w)$ is given while the cross-derivatives do not matter; parts (c) and (d) can be solved only with the partial derivative wrt. the y variable.

(a) [cf. TP1(a) and TP2(b) first question]

- The limit as $x \to 0^+$: the first term of h tends to -2 by inserting $x = 0$. The second term becomes a «$\frac{0}{0}$» form (it is *essential* to check and claim applicability of l'Hôpital's rule!). Using l'Hôpital's rule, we get $\lim_{x\to 0^+} \frac{2x/(1+x^2)}{2x} = \lim_{x\to 0^+} \frac{1}{1+x^2} = 1$. So h tends to $-2 + 1 = -1$ as it should.

- For the limit as $x \to +\infty$, both terms are «$\frac{\infty}{\infty}$». The second term again becomes the limit of $(x^2+1)^{-1}$, but which now tends to 0. The first term tends to 0 too, either by l'Hôpital or by $\frac{x-1}{x^2+1} = \frac{1-1/x}{x+1/x} \to \frac{1-0}{+\infty-0} = 0$. Answer: $\underline{\underline{0}}$.

(b)

- [cf. TP2(b) (parts)] As $h(0^+) = -1$, we have $h < 0$ for all small enough $x < 0$. To use the *intermediate value theorem* on the (continuous!) function h, calculate $h(1) = 0 + \ln 2 > 0$. Therefore, there exists a zero $w \in (0,1)$.
- [cf. TP1(d)] Note that before applying the second-derivative test, one needs to verify that (w,w) is indeed a stationary point. By part (b) we have $0 = h(w) = f'_x(w,w)$, so we calculate $f'_y(x,y) = (1-e^{y-x})\frac{\ln(1+x^2)}{x}$ which is indeed zero when $x = y$, so we have a stationary point at (w,w).

  For the second-derivative test, we calculate $f''_{yy}(x,y) = -e^{y-x}\frac{\ln(1+x^2)}{x}$ which is < 0 on the domain of f. We are given that $f''_{xx}(w,w) > 0$, and so $f''_{xx}(w,w)f''_{yy}(w,w) < 0$, and a saddle point no matter what $f''_{xy}(w,w)$.

(c)

- [cf. TP1(f)] Rewrite the constraints into $x - y \le 0$ and $1 - x \le 0$. The Lagrangian is $f(x,y) - \lambda(x-y) - \mu(1-x)$, and the Kuhn–Tucker conditions become the following (where they are free to include admissibility as well):

$$
\begin{aligned}
0 &= \Big((x+1)e^{y-x} - y\Big)\frac{\ln(1+x^2)}{x^2} + 2\frac{y-e^{y-x}}{1+x^2} - \lambda + \mu && (1)\\
0 &= (1-e^{y-x})\frac{\ln(1+x^2)}{x} + \lambda && (2)\\
\lambda &\ge 0 \qquad \text{with } \lambda = 0 \text{ if } y > x && (3)\\
\mu &\ge 0 \qquad \text{with } \lambda = 0 \text{ if } x > 1. && (4)
\end{aligned}
$$

- Suppose for contradiction that the conditions hold at a point where $x \ne y$ (i.e. $x < y$). Inserting $\lambda = 0$ (by (3)), the easier expression is (2): $0 = (1-e^{y-x})\frac{\ln(1+x^2)}{x}$ which implies $0 = 1 - e^{y-x}$ as x (and thus $\ln(1+x^2)$) is nonzero. But then $y = x$.

  Alternatively, one can split by λ: if $\lambda > 0$, then $x = y$ by (3). If $\lambda = 0$, (2) yields $y = x$ as above.

(d) [cf. TP1(b)] The piece of theory to consider is the *envelope theorem:* $V'(a)$ can be found by differentiating partially wrt. a (in this case: the second variable of f!) and afterwards inserting for x^*. The derivative is $V'(a) = f'_y(x_*, a) = \underline{\underline{(1-e^{a-x_*})\frac{\ln(1+x_*^2)}{x_*}}}$.