University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2018–11–30 exam

- New this semester: The Faculty of Social Sciences has instructed a 66 percent reduction in resources allocated to creating exam problem and grading guideline. The grading guideline is the first to suffer. This note is not suited as a template for an exam paper, nor is it tailored for future teaching.

  The document reflects what was expected in this particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.

- New spring 2018: restricting calculators to the scientific calculator Casio FX-85EX (as well as a simpler arithmetic one).

- Weighting: at the committee's (and in case of appeals: the new grading committee's) discretion. The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice.

- Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

  The committee (and in case of appeals, the new committee) is free to deviate.

Correction after the exam, 2(c): deleted remarks about constant solution, as the exam question ended up only asking for $x \in (0, e)$.

Problem 1 Let a, b, r be constants, all $\in (0, 1/2)$. The equation system

$$\begin{aligned} (aL^a + bL^b)\sqrt{K} &= wLe^{rt} \\ L^a + L^b &= \sqrt{K}e^{rt} \end{aligned} \tag{S}$$

defines continuously differentiable functions $K = K(t, w)$ and $L = L(t, w)$ around the point where $(K, L, t, w) = (4, 1, 0, 2(a + b))$. (You shall not show this.)

(a) Differentiate the system (i.e., calculate differentials).

(b) Suppose that t increases from 0 to 1, and w increases from $2(a + b)$ to $2(a + b) + h$. Use the differentiated system from part (a) to approximate the change in L. (You must use the differentiated system. You cannot expect score for eliminating $\sqrt{K}$ from (S).)

Solution sketch

(a) Calculating differentials:

$$\begin{aligned} \frac{aL^a + bL^b}{2\sqrt{K}}\,dK + (a^2L^{a-1} + b^2L^{b-1})\sqrt{K}\,dL &= we^{rt}\,dL + rwLe^{rt}\,dt + Le^{rt}\,dw \\ (aL^{a-1} + bL^{b-1})\,dL &= \frac{e^{rt}}{2\sqrt{K}}\,dK + r\sqrt{K}e^{rt}\,dt \end{aligned}$$

(b) Put $K = 4$, $L = 1$, $t = 0$, $w = 2(a + b)$, $dt = 1$ and $dw = h$ to get the system

$$\begin{aligned} \frac{a+b}{4}\,dK + 2(a^2 + b^2)\,dL - 2(a + b)\,dL &= 2(a + b)r + h \\ (a + b)\,dL &= \frac{1}{4}\,dK + 2r \end{aligned}$$

which we solve for dL by eliminating $\frac{1}{4}K = (a + b)dL - 2r$:

$$(a + b)^2\,dL - 2(a + b)r + 2(a^2 + b^2)\,dL - 2(a + b)\,dL = 2(a + b)r + h$$

which simplifies to $(3a^2 + 3b^2 + 2ab - 2a - 2b)dL = h + 4(a + b)r$, so that

$$\Delta L \approx dL = \underline{\underline{\frac{h + 4(a + b)r}{3a^2 + 3b^2 + 2ab - 2a - 2b}}}$$

Problem 2

(a) For each constant $k \neq 0$ (positive or negative!), find the limits

i): $\displaystyle\lim_{x\to 0^+} \frac{1}{x^k \ln x}$, ii): $\displaystyle\lim_{x\to +\infty} \frac{1}{x^k \ln x}$, and iii): $\displaystyle\lim_{x\to 0^+} \frac{1}{x^k (\ln x)^{2018}}$.

(b) Show the following *by antidifferentiation* (there is no score for differentiating the right-hand sides):

i): $$\int_x^1 \frac{1}{u\cdot(1-\ln u)}\,du = \ln\big(1-\ln x\big) \qquad \text{(for } 0<x<e\text{)}$$

ii): $$\tfrac{1}{2}\int e^v \cdot \ln((1+e^v)^2)\,dv = (1+e^v)(\ln(1+e^v)-1)+C.$$

(c) Find the general solution of the differential equation (valid for $x \in (0,e)$):

$$\dot{x} = x\cdot(1-\ln x)\cdot e^t \cdot \ln((1+e^t)^2))$$

Solution sketch:

(a) i): $\ln x \to -\infty$. If $k<0$ then $x^k \to +\infty$, and we get «$1/-\infty$», equals $\underline{\underline{0}}$. If $k>0$, then $x^k \ln x$ is a zero-times-infinity, and $\frac{x^{-k}}{\ln x}$ is a «$\infty/(-\infty)$»; by l'Hôpital's rule, it tends to $-k\lim_{x\to 0^+}\frac{x^{-k-1}}{x^{-1}} = -k\lim_{x\to 0^+} x^{-k} = \underline{\underline{-\infty}}$.
ii): For $k>0$, this is $1/(\infty\cdot\infty) = \underline{\underline{0}}$. For $k<0$, $\frac{x^{-k}}{\ln x}$ is a «∞/∞», and we get $-k\lim x^{-k}$ again; Now, it equals $= |k|\lim x^{|k|}$ and as $x\to\infty$, the answer is $\underline{\underline{\infty}}$.
iii): This is $\big(\lim_{x\to 0^+}\frac{1}{x^\ell \ln x}\big)^{2018}$ where $\ell = k/2018$ has the same sign as k.
Using item i), the answers are $(-\infty)^{2018} = \underline{\underline{+\infty \text{ for } k>0}}$, and $0^{2018} = \underline{\underline{0 \text{ for } k<0}}$.

(b) i): Doing the indefinite integral first, avoiding the need to substitute limits: Put $z = (1-\ln u)$; then $dz = -du/u$. We get $-\int\frac{dz}{z} = D - \ln|z| = D - \ln|1-\ln u|$. Now insert for limits: the definite integral is $-\ln(1-\ln 1) + \ln(1-\ln x)$ (since $0<x<e$), equalling $\ln(1-\ln x)$. (One can substitute in the definite integral if one substitutes for «everything u», including the limits.)
ii): Substitute $y = (1+e^v)$ with $dy = e^v dv$, and so we get $\frac{1}{2}\int \ln(y^2)\,dy$. If one has not already rewritten the ln of a square, then one should certainly do so now, to get $\int \ln y\,dy$, which is $y\ln y - y + C$ (it was on the board in a lecture, and an exam problem given for a seminar required it). Substituting back yields the answer.

(c) Separable, with the exam question ruling out the constant solutions. Separate and integrate: $\int\frac{dx}{x(1-\ln x)} = \int e^t \ln((1+e^t)^2)dt$. Using the antiderivatives from (b), we have that $-\ln(1-\ln x) = Q + 2(1+e^t)(\ln(1+e^t)-1)$. Switch and exponentiate: $1-\ln x = e^{-Q}\cdot\big[\exp(1-\ln(1+e^t)\big]^{2(1+e^t)} = K\cdot\big(\frac{e}{1+e^t}\big)^{2+2e^t}$. So $\ln x = 1-K\cdot\big(\frac{e}{1+e^t}\big)^{2+2e^t}$, where K is an arbitrary positive constant. Solution:

$$\underline{\underline{x = \exp\Big\{1-K\big(\frac{e}{1+e^t}\big)^{2+2e^t}\Big\}, \qquad K>0.}}$$

Problem 3 Let t be a real constant. Let $\mathbf{I}$ be the 4×4 identity matrix, and put

$$\mathbf{A}_t = \begin{pmatrix} 1 & t & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & -t & -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Throughout the problem, the prime symbol denotes matrix transpose.

(a) Calculate each of the following or explain why it does not exist:
i): $\mathbf{A}_t^2$ ii): the determinant $|\mathbf{A}_t'\mathbf{A}_t|$ iii): $\mathbf{I1}(\mathbf{I1})'$ iv): $(\mathbf{A}_t\mathbf{1})'\mathbf{A}_t\mathbf{1}$.
(*Hint:* none of your answers should contradict part (b).)

(b) Use part (a) to show that whenever the inverse $\mathbf{A}_t^{-1}$ exists, it is of the form $s\mathbf{A}_t$ for some real number s.

(c) Solve the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{1}$ when t is such that precisely one solution $\mathbf{x}$ exists.

Recall that if $\mathbf{M}$ is an invertible $n \times n$ matrix, where $n > 1$, then $\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|}\mathbf{C}'$ where $\mathbf{C}$ has elements c_{ij} = the cofactor of element (i, j) of $\mathbf{M}$.

(d) If $|\mathbf{M}| = d$ $(\neq 0)$, what is then the determinant of $\mathbf{C}$?

Solution sketch:

(a) is very much alike a TP question, and like the TP it has the trap that they might get orders wrong in iii) (and calculate $\mathbf{1}'\mathbf{1}$ rather than $\mathbf{1}\mathbf{1}'$).

i): $\begin{pmatrix} 1-4 & t-t & 2t-2t & 2-2 \\ 0 & (-1)^2 - 2\cdot 2 & -2+2 & 0 \\ 0 & -(-2)-2 & -4+1^2 & 0 \\ -2+(-2)\cdot(-1) & -2t-2(-t) & -t-(-t) & -4+(-1)^2 \end{pmatrix} = \underline{\underline{-3\mathbf{I}}}$

ii): $|\mathbf{A}_t'\mathbf{A}_t| = |\mathbf{A}_t|^2 = |\mathbf{A}_t^2| = |-3\mathbf{I}| = (-3)^4 = \underline{\underline{81}}$.

iii): $\mathbf{I1} = \mathbf{1}$, so we get $\mathbf{1}\mathbf{1}'$ which is the 4×4 matrix of ones. (Likely it is OK to write «the 4×4 matrix of ones», as long as it is clear that it is 4×4 and not 1×1.)

iv): This is the dot product of $\mathbf{A}_t\mathbf{1}$ with itself. $\mathbf{A}_t\mathbf{1} = (3+t, -1+2, -2+1, -3-t)'$, which when dotted with itself equals $\underline{\underline{2(t+3)^2+2}}$.

(b) Because $(-\frac{1}{3}\mathbf{A}_t)\mathbf{A}_t = \mathbf{I}$, we have $\mathbf{A}_t^{-1} = \underline{\underline{-\frac{1}{3}\mathbf{A}_t}}$.

(c) The unique solution is $\mathbf{A}_t^{-1}\mathbf{1} = -\frac{1}{3}\mathbf{A}_t\mathbf{1}$. Recycling calculations from (a)iv), this equals $-\frac{1}{3}\left(3+t, -1+2, -2+1, -3-t\right)' = \underline{\underline{\left(-1-\frac{t}{3}, -\frac{1}{3}, \frac{1}{3}, 1+\frac{t}{3}\right)'}}$.

(d) $d \cdot \mathbf{M}^{-1} = \mathbf{C}'$, so $|\mathbf{C}| = |d \cdot \mathbf{M}^{-1}| = d^n/d = \underline{\underline{d^{n-1}}}$.

Problem 4 Let $c > 0$ be a constant. Let u be a continuously differentiable function of two variables. Consider the maximization problem

$$\max\ u(x,y) \quad \text{subject to the constraint} \qquad c - u(1-x, 1-y) = 0 \tag{L}$$

and the nonlinear programming problem

$$\max\ u(x,y) \quad \text{subject to} \qquad c - u(1-x,1-y) \le 0, \quad 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1 \tag{K}$$

(Note: «$0 \le x \le 1$» forms two constraints $x \ge 0$ and $x \le 1$, and similar for «$0 \le y \le 1$».)

(a) i): State the Lagrange conditions associated with problem (L), and state the Kuhn–Tucker conditions associated with problem (K).
(Possible hint if the «u» in a constraint is confusing: write first as $c - g(x,y) = 0$ (resp. ≤ 0), and insert afterwards, so that you in the end get conditions with derivatives of only u, not of «g».)

ii): True or false? *«The point $(x,y) = (\frac{1}{2}, \frac{1}{2})$ will satisfy the Lagrange conditions associated with problem (L), as long as the constraint holds.»*
(Do not expect score for an unsubstantiated guess!)

(b) Let in this part $u(x,y) = 2(e + x) - e^{2(1-x)} - (1+e)e^{1-2y}$ and $c = 0$.

i): Show that the point $(x,y) = (\frac{1}{2}, \frac{1}{2})$ is indeed optimal for problem (K).
(If unable to do (K), then for partial score: show optimality for (L) instead.)

ii): If c is decreased from 0 to -0.03, approximately how much does the optimal value change? (You can take the optimality from part i) for granted regardless of whether you managed to show it. If you did problem (L) in i), you can consider (L) here too.)

Solution sketch Let $L(x,y) = u(x,y) - \lambda \cdot (c - u(1-x, 1-y))$ and let $K(x,y) = L(x,y) + \alpha x + \beta y - \gamma(x-1) - \delta(y-1)$ be the respective Lagrangians. The first partial derivatives of L are $u'_x(x,y) - \lambda u'_x(1-x,1-y)$ and $u'_y(x,y) - \lambda u'_y(1-x,1-y)$. (For K: will follow.)

(a) i): Lagrange conditions: $u'_x(x,y) = \lambda u'_x(1-x,1-y)$, $u'_y(x,y) = \lambda u'_y(1-x,1-y)$ and the constraint $u(1-x,1-y) = c$.

Kuhn–Tucker conditions:

$$\begin{aligned}
&0 = u'_x(x,y) - \lambda u'_x(1-x,1-y) + \alpha - \gamma \\
&0 = u'_y(x,y) - \lambda u'_y(1-x,1-y) + \beta - \delta \\
&\lambda \ge 0 \quad \text{with } \lambda = 0 \text{ if } u(1-x,1-y) > c \\
&\alpha \ge 0 \quad \text{with } \alpha = 0 \text{ if } x > 0 \\
&\beta \ge 0 \quad \text{with } \beta = 0 \text{ if } y > 0 \\
&\gamma \ge 0 \quad \text{with } \gamma = 0 \text{ if } x < 1 \\
&\delta \ge 0 \quad \text{with } \delta = 0 \text{ if } y < 1
\end{aligned}$$

They are free to include admissibility or not. Also, they can use alternative formulations, for example

$$u'_x(x,y) - \lambda u'_x(1-x,1-y) \begin{cases} \leq 0 & \text{if } x = 0 \\ \geq 0 & \text{if } x = 1 \\ = 0 & \text{if } x \in (0,1) \end{cases}$$

$$u'_y(x,y) - \lambda u'_y(1-x,1-y) \begin{cases} \leq 0 & \text{if } y = 0 \\ \geq 0 & \text{if } y = 1 \\ = 0 & \text{if } y \in (0,1) \end{cases}$$

$$\lambda \geq 0 \quad \text{with } \lambda = 0 \text{ if } u(1-x,1-y) > c$$

ii): True: Putting $x = y = 1/2$, conditions say $u'_x(\frac{1}{2},\frac{1}{2}) = \lambda u'_x(\frac{1}{2},\frac{1}{2})$, $u'_y(\frac{1}{2},\frac{1}{2}) = \lambda u'_y(\frac{1}{2},\frac{1}{2})$ and $u(\frac{1}{2},\frac{1}{2}) = c$; the two first hold (with $\lambda = 1$), so it is OK iff the constraint holds.

(b) i): We have $u(\frac{1}{2},\frac{1}{2}) = 2e + 1 - e - (1+e)e^0 = 0$, so that constraint is active. The others are not. The Kuhn–Tucker conditions hold with $\lambda = 1$ and all other multipliers vanishing. Since this u is concave – being a sum of concave functions – we have found the solution.

(That argument goes for problem (L) as well.)

Note that the extreme value theorem is not sufficient – it only shows existence of a solution, but not that there exists only one point that satisfies the conditions. It might be acceptable – depending on how the argument is done – to draw an Edgeworth box and making a graphical argument about single-touching («convex sets» terminology is not curriculum).

ii): $\lambda = 1$, so the change is ≈ 0.03. I suggest not to stress the sign, as the usual formulation in this course would be the opposite sign.