ECON3120/4120 Mathematics 2: on the 2019–11-28 exam

- *Standard disclaimer:* A note like this is not suited as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- Weighting: At the discretion of the committee (and in case of appeals: the new grading committee). The committee might want to consider the next two bullet items.

  The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and this has been communicated.
- *Special considerations for 2019: new exam format.* Starting 2019, the exam is 4 hours (changed from 3); one can no longer bring written support material (instead, there is a pre-announced known «Rules and formulas» attachment following the problem set); and, the problem set is now in English only (however there is no change in the regulations as for what languages are permitted for the submitted papers).

  As the Department decided against discussing more extensive changes, and so the format should not be taken to intend changes in overall requirements. Rather, there is a hope that this will facilitate better differentiation between candidates; the grading distribution may of course be affected, should the committee find it appropriate.

  Given that this is the first exam set in the new format, it might set standards for the years to come, and the committee should set benchmarks with caution. There might be less reason to stress the percentage-to-grade tables that have been applied earlier (nominally defaulting to 91–75–55–45–40); the 40 percent pass mark does however hold a long history and as a preliminary view I would consider it to be more of a constant than the other thresholds.
- *2019 change in compulsory activities:* The level of compulsory activities required in order to sit in on the exam, have increased from one handed-in term paper (not part of the eleven seminar assignments) to three (three approved out of four sets assigned*, all four now part of the eleven seminar assignments). Some questions in this exam follow term paper problems very closely, and could for that reason be considered – everything else equal – to be «easier» or «should certainly be very well known». In this document, the abbreviations «`TPM§N`» refers to task `N` of this semester's `M`th compulsory term paper problem set. The document also gives partial indications on how the topics have been covered in the other seven seminar assignments (which are publicly available†).
- *2019 change in teaching format:* The changes in teaching format might affect the students' chances at learning. This course had an intensive beginning with eleven lectures by September 5th, most of which being optimization. Although other courses typically taken in the same semester had delayed their start-up, the intensive format might have been challenging, and topics perceived as minor might have been «dwarfed». A number of videos have been produced for the students to be able to catch up (published to enrolled students only), and the seminar problem sets – and in particular the hand-ins – have attempted to pick up on topics that might have been lost in the load of new information. The handed-in term papers might suggest that limits and l'Hôpital's rule could have suffered, and might come across as a de facto harder topic than earlier. Furthermore, the envelope theorem is no longer known from the new Mathematics 1 course ECON1100.

*as before, not including a make-up attempt – which in any case only applied to a very few candidates

†Although the four hand-in problem sets are not published in the open, they have been communicated to the committee when a draft exam problem set was sent for review.

Addendum after grading: Apparently, the problem set was on the easy side relative to the exam format. Grading would be based on the 92 – 77 – 58 – 46 – 40 thresholds once recommended by the Norwegian Mathematical Council (commonly applied at The Faculty of Mathematics and Natural Sciences).

Problems (restated as given) and solutions and annotations (boxed) follow:

Problem 1 Take for granted that the following equation system determines K and L as continuously differentiable functions of (s, t) near the point where $(s, t, K, L) = (0, 2, 1, 0)$:

$$tK + \ln K + \ln(1 + L) + L^2 = 2$$
$$sL + K^2 + e^{KL} - t = 0$$

(a) Differentiate the system (i.e., calculate differentials).

(b) Calculate $\frac{\partial K}{\partial t}(0, 2)$ and $\frac{\partial L}{\partial t}(0, 2)$.

Problem 1 solved:

(a) Differentiating:

$$t\,dK + K\,dt + \frac{1}{K}\,dK + \frac{1}{L+1}\,dL + 2L\,dL = 0$$
$$s\,dL + L\,ds + 2K\,dK + e^{KL}(L\,dK + K\,dL) - \ dt = 0$$

(b) Insert for $(s, t, K, L) = (0, 2, 1, 0)$:

$$0 = 2\,dK + 1\,dt + 1\,dK + 1\,dL + 0\,dL \qquad \text{so} \qquad 3\,dK + dL + dt = 0$$
$$0 = 0\,dL + 0\,ds + 2\,dK + e^0(0\,dK + 1\,dL) - \ dt \qquad \text{so} \qquad 2\,dK + dL - dt = 0$$

Subtracting equations, $dK = -2\,dt$ which also yields $dL = dt - 2\,dK = (1 + 4)\,dt$. Therefore, $\underline{\underline{\frac{\partial K}{\partial t}(0, 2) = -2}}$ and $\underline{\underline{\frac{\partial L}{\partial t}(0, 2) = 5}}$

Problem 1 notes: *Although this topic was covered in just slightly more than a double lecture near the end of the course, it has been communicated that it has appeared in ≈ half of the exams the last years. A total of four exam problems with differentiation in equation system were assigned for the last two seminars, and yet another has a video walkthrough.*
Part (a) above calculates term by term (like Wolfram Alpha does here). That is OK, one need not collect terms.

Problem 2 Let $\mathbf{A} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} t & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} t & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} -t \\ 2 \\ 3 \end{pmatrix}$

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{d}$ depend on the constant t (real number).
Do not select a value for t; in particular, the answer to (a) will be a t-dependent matrix.

(a) Among the matrix products $\mathbf{ABd}$, $\mathbf{BCd}$, $\mathbf{B}^2$, $\mathbf{C}^2$ and $\mathbf{d}^2$, pick *one* that is well-defined and calculate it (for every t).
(You can pick one you find easy to calculate. A harder one is not worth higher score.)

(b) For each of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{d}$: calculate its determinant or point out that it does not exist.

(c) Show that for every real t, the equation system $\mathbf{Cx} = \mathbf{d}$ has at least one solution $\mathbf{x}$.
(You are not asked to solve completely, but you are allowed to solve as far as you need in order to answer the question.)

Problem 2 solved:

(a) Picking $\mathbf{C}^2$:

$$\mathbf{C}^2 = \begin{pmatrix} t & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} t & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} t^2+2 & 2t-2 & 2 \\ t-1 & 2+1+1 & -1+1 \\ 1 & -1+1 & 1+1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} t^2+2 & 2t-2 & 2 \\ t-1 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}}}$$

(b) <u>No determinant is defined for $\mathbf{B}$ and $\mathbf{d}$</u> as they are not square. $|\mathbf{A}| = 1 - (-t^2) = \underline{\underline{1 + t^2}}$ and

$$|\mathbf{C}| = t\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 = \underline{\underline{-2t-2}}$$

(c) *Unique* solution exists iff $0 \neq |\mathbf{C}|$ i.e. iff $t \neq -1$. Only the case $t = -1$ remains:

$$\left(\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \begin{matrix} \urcorner \\ \leftarrow\!\lrcorner_+ \\ \end{matrix} \sim \left(\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right) \begin{matrix} \\ \urcorner^{-1} \\ \leftarrow\!\lrcorner_+ \end{matrix}$$

Last equation: $0 = 0$, delete it. Choose e.g. x_2 free, there is indeed an x_1 (determined by eq. #1) and an x_3 (determined by eq. #2).

Problem 2 notes: *The last linear algebra lecture concerned specifically equation systems with parameter (part (c)), and it was assigned for the two last seminars – however, it was not part of the four ordinary term paper sets, as linear algebra was lectured late in the semester. Neither was determinants. (Matrix products and Gaussian elimination were assigned for hand-ins, TP4§4a and §4c).*

It seems to have stuck to Mathematics 2 in the more recent years, that calculating matrix products merits score by itself. It used not to be the case. Problem (a) has a potential trap in inviting those who mistake element-wise multiplications for matrix products, to actually get an answer (without wasting much time though, selecting $\mathbf{d}^2$*). Problem (b) has a slight twist compared to most similar problems: it asks to point out that only square matrices have well-defined determinant.*

«Any reasonable» notation for vectors or matrices is allowed. This semester, the lectures used overarrow notation not only for vectors (minuscle) but even for matrices (capitals), e.g. $\vec{C}\vec{x} = \vec{d}$*, and where vectors default to columns.*

Problem-specific notes:

(a) *The problem does in part invite those who mistake element-wise multiplications for matrix products, to select for example* $\mathbf{d}^2$ *(and not spend too much time).*

- $\mathbf{B}^2$ *and* $\mathbf{d}^2$ *are undefined. The well-defined matrix products can be found here:* $\mathbf{ABd}$ *;* $\mathbf{BCd}$ *;* $\mathbf{C}^2$ *(note for notation: in Math 2 one is encouraged not to use dots for matrix products).*
- *The above solution surely shows enough calculations to justify the answer.*

(b) *Exercise judgement when the nonexistence of* $|\mathbf{B}|$ *and* $|\mathbf{d}|$ *is stated without justification. Some might find that obvious.*

The above solution cofactor expands $\mathbf{C}$ *along the first row. It was intentional to give a matrix where any choice of row or column would lead to calculation of (at least) two cofactors.*

(c)
- *They are allowed to delete a zero row despite that, strictly speaking, is typically not part of a textbook definition of Gaussian elimination. Thus, the above solution could continue*

$$\sim \left(\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

even though the latter is not row-equivalent; that concept – even the phrase – is not required knowledge, and neither is row space. In Mathematics 2 lingo the $\sim$ *means they represent equivalent equation systems.*

- *Also, notation: that vertical separator between the LHS and RHS has been advocated, although not being part of the curriculum.*
- *Part (c) can be solved without part (b), by solving with* t *left general. Note, it has been stressed repeatedly – at unknown avail, judging from subsequent hand-ins – that division by zero is not allowed, and it would be wrong to scale the first equation by* $\frac{1}{t}$ *even if that were to be undone later. The tip to avoid this by pushing the* t *into the future, is lectured. A solution could go e.g. as follows (or, with some additional operations that interchange rows):*

$$\left(\begin{array}{ccc|c} t & 2 & 0 & -t \\ 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \begin{array}{l} \leftarrow + \\ \rfloor (-t) \\ \end{array} \sim \left(\begin{array}{ccc|c} 0 & 2+t & -t & -3t \\ 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \begin{array}{l} \leftarrow + \\ \leftarrow + \\ \rfloor -(2+t) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 0 & 0 & -2t-2 & -6t-6 \\ 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & t+1 & 3(t+1) \end{array}\right)$$

Iff $t+1 \neq 0$*, the last equation is equivalent to* $x_3 = 3$*; solving bottom–up, there will be a solution* $x_2 = 0$ *and* $x_1 = -1$*. But* $(-1, 0, 3)'$ *is a solution when* $t = -1$ *as well.*

Problem 3

(a) Show *by antidifferentiation* that $\int te^{-t/2}\,dt = C - 2(t+2)e^{-t/2}$.
(There is no score for differentiating the right-hand side.)

(b) For the differential equation $\dot{x} = \dfrac{e^x - 1/e}{e^x}\, te^{-t/2}$, find the following particular solutions:

- the one satisfying $x(-2) = -1$
- the one satisfying $x(-2) = 1$.

(c) Use the substitution $u = \ln z$ to calculate $\int_1^\infty \dfrac{\ln z}{z^{3/2}}\,dz$.
For full score, you must use this substitution. You can get partial score by using other methods.

Problem 3 solved:

(a) $\int \underbrace{t}_{f}\underbrace{e^{-t/2}}_{g'}\,dt = t\cdot\underbrace{(-2)e^{-t/2}}_{g} - \int(-2)e^{-t/2}\,dt = C - 2te^{-t/2} - 4e^{-t/2}$, *OK!*

(b) $e^x - 1/e$ is zero when $x = -1$, so the first particular solution is the constant $\underline{\underline{x(t) \equiv -1}}$.

For the other: $\dfrac{e^x}{e^x - 1/e}dx = te^{-t/2}dt$. For the LHS, use $w = e^x - 1/e$, $dw = e^x\,dx$ to get $\int \dfrac{dw}{w} = C - 2(t+2)e^{-t/2}$ (by (a)). At $t = -2$, $x = 1$ and $w = e^1 - 1/e > 0$ so $\ln|w| = \ln w$ and $\ln w = \ln(e^x - 1/e) = C - 2(t+2)e^{-t/2}$, and $\ln(e - 1/e) = C - 0$. This gives

$$e^x - 1/e = e^{\ln(e-1/e)}\cdot e^{-2(t+2)e^{-t/2}} \implies \underline{\underline{x = \ln\left(1/e + (e-1/e)e^{-2(t+2)e^{-t/2}}\right)}}$$

(c) $u = \ln z \Rightarrow du = \frac{1}{z}dz$, so $\frac{\ln z}{z}dz = u\,du$. Need $z^{-1/2} = (e^u)^{-1/2} = e^{-u/2}$, so

$$\int \frac{\ln z}{z^{3/2}}\,dz = \int ue^{-u/2}du \overset{(a)}{=} C - 2(u+2)e^{-u/2} = C - 2(2+\ln z)z^{-1/2}, \quad \text{and}$$

$$\int_1^\infty \frac{\ln z}{z^{3/2}}\,dz = \lim_{b\to+\infty}\Big[-2(2+\ln z)z^{-1/2}\Big]_{z=1}^{b} = 4 - 2\underbrace{\lim_{b\to+\infty}\frac{2+\ln b}{b^{1/2}}}_{«\infty/\infty»}$$

$$= 4 - 2\lim_{b\to+\infty}\frac{b^{-1}}{\frac{1}{2}b^{-1/2}} = 4 - 4\lim_{b\to+\infty} b^{-1/2} = \underline{\underline{4}}.$$

Problem 3 notes: *Problem 3 is a near-blueprint of a subset of TP4.*

Concerning (b): To «find» a solution in this course, has generally been taken to include solving (algebraically) for x =[only t, no «x»]. There is no need to look for simpler forms than above. Three questions in two compulsory term papers (the others being TP3§§fg) had separable differential equations where one had to find the constant solution. Hopefully there will be larger aversion against dividing without checking for zero than midway through the course.

Concerning (c): For definite integrals requiring substitution, it has been encouraged to do indefinite integrals first, unless knows perfectly well how to substitute limits. Also, the integral is improper and should be treated as such. The underbrace is sufficient justification for applying l'Hôpital.

Problem 4 Define the C^1 function $h(t) = \dfrac{e^{2qt} - 2qt + t^q}{\ln(1+t^2)}$ for all $t > 0$.
Here, $q \in (0,1)$ is a constant.

(a) Show that $\lim_{t\to 0^+} h(t)$ and $\lim_{t\to+\infty} h(t)$ both diverge to $+\infty$, for every $q \in (0,1)$.
(*Hint:* For one of these limits, it might be useful that $h(t) = \dfrac{e^{2qt}}{\ln(1+t^2)} \cdot \Big(1 + \dfrac{t^q - 2qt}{e^{2qt}}\Big)$.)

(b) From part (a) it follows that $h'(t_1) < 0$ for some t_1 near 0, and that $h'(t_2) > 0$ for some large t_2. (You are not asked to show this.)
Use this to show that h has at least one stationary point t_*. (Do not attempt to *find* t_*!)

(c) Take for granted that t_* minimizes h. The minimum value $V = h(t_*)$ depends on q. Find an expression for $V'(q)$.

Problem 4 solved:

(a) $\lim_{t\to 0^+} h(t) =«\frac{1+0+0^q}{0^+}»$ is positive infinity. For $\lim_{t\to+\infty} h(t)$, use the form given to get $+\infty$:

$$\lim_{t\to+\infty} \overbrace{\frac{e^{2qt}}{\ln(1+t^2)}}^{«\infty/\infty»} \cdot \underbrace{\Big(1 + \overbrace{\frac{t^q - 2qt}{e^{2qt}}}^{\text{exp decay vs power}}\Big)}_{\therefore,\ \to 1\ (\neq 0)} = 1 \cdot \lim_{t\to+\infty} \frac{2qe^{2qt}}{2t/(1+t^2)} = q \lim_{t\to+\infty} (1+t^2)\ \overbrace{\frac{e^{2qt}}{t}}^{\text{exp vs. } t}$$

(b) $h'(t_1) < 0 < h'(t_2)$, and by the intermediate value theorem, h' hits zero (i.e. h is stationary) at some t_* between t_1 and t_2.

(c) The envelope theorem:

$$V'(q) = \frac{\partial}{\partial q}\Big[\frac{e^{2qt} - 2qt + t^q}{\ln(1+t^2)}\Big]\Big|_{t=t_*} = \underline{\underline{\frac{1}{\ln(1+t_*^2)} \cdot \Big(2t_* \cdot e^{2qt_*} - 2t_* + t_*^q \ln t_*\Big)}}$$

Problem 4 notes: *Both TP2§§abd and TP4§1 had a function with a parameter, and asked for both limits, application of the intermediate value theorem and the envelope theorem. As follow-up to TP2 it was assigned for the next seminar too (exam autumn 2017 parts 1abd).*

Specific notes:

(a) *They are allowed to «know» that exponential decay kills polynomial growth like indicated, but unfortunately that rule is often abused, especially when there are two exponential terms (or, for that matter, if there is a log in the exponent); exercise judgement.*

Also often abused: arithmetic of the reals, treating infinity as an ordinary number.

(b) *This part tests a single piece of theory. The «Do not attempt» is not only intended to clarify the question, but also a hint that finding is not likely to be a successful method. The hard parts of (b) are likely to use the intermediate value theorem on a function that is something's derivative, and to keep the intermediate value theorem from the extreme value theorem. Note, one is not required to justify the existence and continuity of h'.*

(c) *Abuse of notation like $\frac{\partial}{\partial q}h$ occurs even in the scientific literature, so stress content over form here. It was intentional to test differentiation wrt. an exponent when base number $\neq e$.*

Problem 5 Consider the problem

$$\max\ y^2 + (x-1)y \qquad \text{subject to} \qquad x^2 + 18y \le 45, \quad x \ge 2, \quad y \ge 1/2 \tag{K}$$

(a)
- State the associated Kuhn–Tucker conditions, and
- show that some multiplier must be $\neq 0$ for these conditions to be satisfied at an admissible point (x, y). ("Admissible": that satisfies the three constraints.)

(b) Are the Kuhn–Tucker conditions satisfied at
- the point $(x_1, y_1) = (3, 2)$?
- the point $(x_2, y_2) = (6,\ 1/2)$?

Problem 5 solved:

(a)
- Let $L(x, y) = y^2 + (x-1)y - \lambda(x^2 + 18y - 45) - \alpha(2 - x) - \beta(\frac{1}{2} - y)$. Conditions:

$$\begin{array}{lll} 0 = y - \lambda \cdot 2x + \alpha & & \text{(FOCx)} \\ 0 = 2y + x - 1 - \lambda \cdot 18 + \beta & & \text{(FOCy)} \\ \lambda \ge 0 & \text{and if } x^2 + 18y < 45\text{: } \lambda = 0 & (\Lambda) \\ \alpha \ge 0 & \text{and if } x > 2\text{: } \alpha = 0 & \text{(A)} \\ \beta \ge 0 & \text{and if } y > \frac{1}{2}\text{: } \beta = 0 & \text{(B)} \end{array}$$

- If all multipliers were 0, condition (FOCx) would say $y = 0$ which is not admissible.

(b)
- At point (x_1, y_1), we have $x_1^2 + 18y_1 = 45$, while the two other constraints are inactive so $\alpha = \beta = 0$. Inserting, the conditions reduce to:

$$0 = 2 - 6\lambda + 0 \qquad 0 = 4 + 3 - 1 - 18\lambda \qquad \lambda \ge 0$$

which hold true with $\lambda = 1/3$.

- At point (x_2, y_2), $x_2 = 6 > 2$, so $\alpha = 0$; but $x_1^2 + 18y_1 = 36 + 9 = 45$ and $y_2 = 1/2$ are active. Conditions reduce to:

$$0 = \tfrac{1}{2} - \lambda \cdot 2 \cdot 6 \qquad 0 = 1 + 6 - 1 - 18\lambda + \beta \qquad \lambda \ge 0 \qquad \beta \ge 0$$

So $\lambda = \frac{1}{24}$ from the first of these; the second then gives $\beta = 18\lambda - 6 = \frac{3}{4} - 6 < 0$, contradicting nonnegativity. Conditions fail at this point.

Problem 5 notes: *TP4§2b and most of TP1 was Lagrange/Kuhn–Tucker, and Kuhn–Tucker recurred in an exam problem assigned for the very last seminar. All of these had a «test a given point» type problem, so one should be familiar with the method of inserting points and extracting possible multiplier values. The conditions failing at a point might be the less familiar element.*

(a) *Equivalent formulations like $\alpha \ge 0 = \alpha(x-2)$ are perfectly fine. In case anyone should want to use the formulation $y - \lambda \cdot 2x \le 0$ $\quad(= 0$ if $x > 2)$ etc., there will still have to be a nonzero multiplier as, in fact, λ must be > 0.*

Admissibility can be included or not in the conditions, their choice. (That is also the reason why admissibility is mentioned in the second bullet item of the problem.)

It has been stressed that the FOCs should have the derivatives written out. Merely «$0 = L'_x$» etc., will not be sufficient.

(b) *One can of course remark initially that both points have* $x > 2$, *so* $\alpha = 0$, *and proceed from there. Indeed, for* (x_2, y_2) *then* $\beta = 18\lambda - 6 = \frac{3}{2} - 6 < 0$ *(impossible!), without addressing whether any other constraint is active. The above solution could be shorter for* (x_2, y_2), *but is aligned more to likely actual answers.*

For (x_1, y_1), *one shall check* $x^2 + 18y = 45$ *to validate the positive* λ.