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Solutions of compulsory problem set 1
ECON3120/4120 Mathematics 2, spring 2004

Problem 1

(a) $f'(x) = \dfrac{9e^x(e^x+1) - 9e^x e^x}{(e^x+1)^2} - 2 = \dfrac{-2e^{2x} + 5e^x - 2}{(e^x+1)^2}$. Thus,

$$(*) \qquad f'(x) = 0 \iff e^{2x} - \tfrac{5}{2}\, e^x + 1 = 0.$$

Let $z = e^x$. Then $(*)$ is equivalent to the quadratic equation $z^2 - \frac{5}{2}z + 1 = 0$, which has the roots $z_1 = \frac{1}{2}$, $z_2 = 2$. Hence, the stationary points of f are

$$x_1 = \ln z_1 = \ln \tfrac{1}{2} = -\ln 2 \ (\approx -0.6931), \quad x_2 = \ln z_2 = \ln 2 \ (\approx 0.6931).$$

(b) f is differentiable everywhere, so the only possible local extreme points of f are x_1 and x_2. A little calculation shows that the second derivative of f is

$$f''(x) = \frac{9e^x - 9e^{2x}}{(e^x+1)^3}.$$

The values of f'' at the stationary points are

$$f''(x_1) = \frac{9/2 - 9/4}{(1/2+1)^3} = \frac{9/4}{27/8} = \frac{2}{3} > 0, \quad f''(x_2) = \frac{18-36}{(2+1)^3} = -\frac{18}{27} = -\frac{2}{3} < 0.$$

It follows that x_1 is a local minimum point of f, and x_2 is a local maximum point.

(c) For all x, we have $0 < \dfrac{9e^x}{e^x+1} < \dfrac{9e^x}{e^x} = 9$. It is therefore easy to see that

$$\lim_{x\to\infty} f(x) = -\infty, \qquad \lim_{x\to-\infty} f(x) = \infty.$$

It follows that f has no global maximum or minimum.

(d) From part (a) we get $f'(x) = -\dfrac{2(e^x - \frac{1}{2})(e^x-2)}{(e^x+1)^2} = -\dfrac{2(e^x - e^{x_1})(e^x - e^{x_2})}{(e^x+1)^2}$.
It follows that

$$f'(x) \begin{cases} < 0 & \text{if } x < x_1 = -\ln 2, \\ > 0 & \text{if } x \in (x_1, x_2) = (-\ln 2, \ln 2), \\ < 0 & \text{if } x > x_2 = \ln 2. \end{cases}$$

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Hence, f is strictly decreasing in $(-\infty, x_1] = (-\infty, -\ln 2]$, strictly increasing in $[x_1, x_2] = [-\ln 2, \ln 2]$, and strictly decreasing in $[x_2, \infty) = [\ln 2, \infty)$. This shows that f can have at most 3 zeros, one in each of these three intervals.

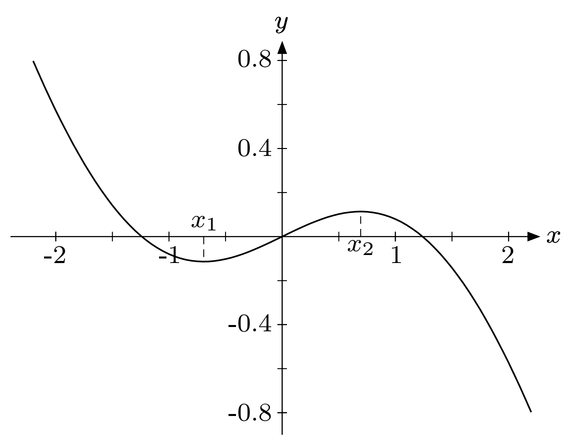
Now, $f(-3) = \dfrac{9e^{-3}}{e^{-3}+1} + (6 - \frac{9}{2}) > 0$, $f(x_1) = f(-\ln 2) = -\frac{3}{2} + 2\ln 2 < 0$, $f(x_2) = f(\ln 2) = \frac{3}{2} - 2\ln 2 > 0$, and $f(3) = \dfrac{9e^{3}}{e^{3}+1} - 6 - \frac{9}{2} < 9 - 6 - \frac{9}{2} < 0$.

(With a pocket calculator we get $f(-3) \approx 1.9268$, $f(x_1) \approx -0.1137$, $f(x_2) \approx 0.1137$, $f(3) \approx -1.9268$.)

The intermediate value theorem ("skjæringssetningen") then shows that there is indeed a zero of f in each of the three intervals $(-3, x_1)$, (x_1, x_2), $(x_2, 3)$.

Comment: The function values given above may have led you to suspect that $f(-x) = -f(x)$. This is indeed true, and the easiest way to show this is probably to show $f(-x) + f(x) = 0$ by direct calculation. One consequence of $f(-x) = -f(x)$ is that $f(0) = 0$, so $x = 0$ is one of the solutions of $f(x) = 0$. The other two zeros of f are $\approx \pm 1.2402$.

The figure shows the graph of f. Note that there are different scales on the two axes.



Problem 1. The graph of f

Problem 2

(a) The determinant of the coefficient matrix is

$$|\mathbf{A}| = \begin{vmatrix} 1 & k & 3 \\ k & 9 & k \\ 7 & k & 5 \end{vmatrix} = 4k^2 - 144 = 4(k^2 - 36) = 4(k+6)(k-6).$$

By Cramer's rule the system has a unique solution if $k \neq \pm 6$.

If $k = 6$, the system becomes

$$\begin{aligned} x + 6y + 3z &= 2 \\ 6x + 9y + 6z &= 3 \\ 7x + 6y + 5z &= -6 \end{aligned}$$

which has no solutions. (Gaussian elimination gives

$$\begin{pmatrix} 1 & 6 & 3 & 2 \\ 6 & 9 & 6 & 3 \\ 7 & 6 & 5 & -6 \end{pmatrix} \begin{matrix} -6 \quad -7 \\ \longleftarrow\!\lrcorner \quad | \\ \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!\lrcorner \end{matrix} \sim \begin{pmatrix} 1 & 6 & 3 & 2 \\ 0 & -27 & -12 & -9 \\ 0 & -36 & -16 & -20 \end{pmatrix} \begin{matrix} -4/3 \\ \longleftarrow\!\lrcorner \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 6 & 3 & 2 \\ 0 & -27 & -12 & -9 \\ 0 & 0 & 0 & -8 \end{pmatrix},$$

and the last row of the final matrix represents the equation $0 = -8$, which obviously has no solution.)

If $k = -6$, the system becomes

$$\begin{aligned} x - 6y + 3z &= 2 \\ -6x + 9y - 6z &= 3 \\ 7x - 6y + 5z &= -6 \end{aligned}$$

By Gaussian elimination or otherwise, we get the solution

$$x = -\frac{1}{3}t - \frac{4}{3}, \quad y = \frac{4}{9}t - \frac{5}{9}, \quad z = t, \qquad t \text{ arbitrary},$$

with one degree of freedom.

(b) The cases $k = \pm 6$ were dealt with in part (a). For $k \neq \pm 6$, we can use Cramer's rule. We shall need the three determinants

$$D_1 = \begin{vmatrix} 2 & k & 3 \\ 3 & 9 & k \\ -6 & k & 5 \end{vmatrix} = -8k^2 - 6k + 252 = 2(k+6)(21-4k),$$

$$D_2 = \begin{vmatrix} 1 & 2 & 3 \\ k & 3 & k \\ 7 & -6 & 5 \end{vmatrix} = -8k - 48 = -8(k+6),$$

$$D_3 = \begin{vmatrix} 1 & k & 2 \\ k & 9 & 3 \\ 7 & k & -6 \end{vmatrix} = 8k^2 + 18k - 180 = 2(k+6)(4k-15).$$

Cramer's formula then gives

$$x = \frac{D_1}{|\mathbf{A}|} = \frac{21-4k}{2k-12}, \quad y = \frac{D_2}{|\mathbf{A}|} = \frac{2}{6-k}, \quad z = \frac{D_3}{|\mathbf{A}|} = \frac{4k-15}{2k-12}.$$

Problem 3

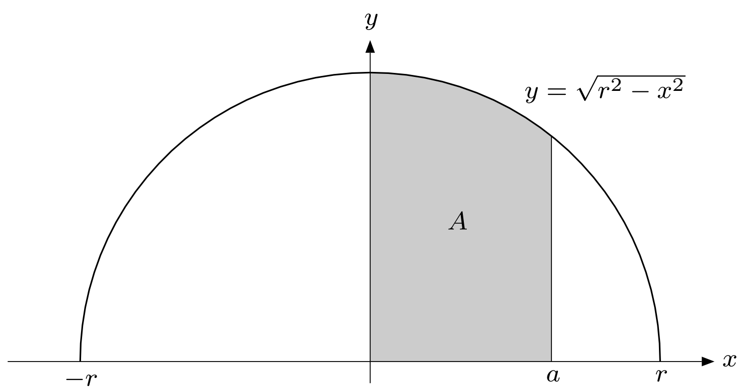
(a)

(i) $$\int \frac{x^3+2}{x}\,dx = \int \Big(x^2 + \frac{2}{x}\Big)dx = \frac{x^3}{3} + 2\ln|x| + C = \frac{x^3}{3} + \ln x^2 + C.$$

(ii) With the substitution $u = 1 - x$ we get $du = -dx$ and

$$\int_{(x=)\,0}^{(x=)\,1} \sqrt{1-x}\;dx = -\int_{(u=)\,1}^{(u=)\,0} \sqrt{u}\;du = \int_0^1 u^{1/2}\,du = \Big|_0^1 \frac{2}{3}u^{3/2} = \frac{2}{3}\,.$$

(b) The integral $\int_0^a \sqrt{r^2-x^2}\,dx$ gives the area of the region A between the graph of $y = \sqrt{r^2-x^2}$ and the x-axis over the interval $[0,a]$.



$\int_0^a \sqrt{r^2-x^2}\,dx$ is the area of the shaded region.

With $y = \sqrt{r^2-x^2}$ we get $x^2 + y^2 = r^2$, so the graph of $y = \sqrt{r^2-x^2}$ is part of the circle with center at the origin and radius r. When $a = r$, the shaded region becomes one quarter of the whole circular disk, and therefore its area equals $\frac{1}{4}\pi r^2$.

(c) Integrations by parts:

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}}\,dx &= \int \ln x \cdot \frac{1}{\sqrt{x}}\,dx = \ln x \cdot 2\sqrt{x} - \int \frac{1}{x}\cdot 2\sqrt{x}\,dx \\ &= 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C.\end{aligned}$$

Problem 4

L'Hôpital's rule:

(i) $$\lim_{x\to 0} \frac{\ln(e^x + x)}{x} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 0} \frac{\frac{e^x+1}{e^x+x}}{1} = \lim_{x\to 0} \frac{e^x+1}{e^x+x} = \frac{1+1}{1+0} = 2.$$

$$\text{(ii)} \quad \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} = \text{“}\frac{\infty}{\infty}\text{”} = \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} = \text{“}\frac{\infty}{\infty}\text{”}$$

$$= \lim_{x\to\infty} \frac{e^x}{e^x + 1} = \lim_{x\to\infty} \frac{1}{1 + e^{-x}} = \frac{1}{1+0} = 1.$$

(We could have avoided the second application of l'Hôpital's rule by using

$$\frac{e^x + 1}{e^x + x} = \frac{1 + e^{-x}}{1 + xe^{-x}}$$

and remembering that $\lim_{x\to\infty} xe^{-x} = 0$. On the other hand, we could have used l'Hôpital's rule a third time to evaluate $\lim_{x\to\infty} \frac{e^x}{e^x + 1}$ as an “$\frac{\infty}{\infty}$” expression.)

Problem 5

$$F'(T) = \frac{d}{dT} \int_0^T f(t)e^{-rt}\,dt + S'(T)e^{-rT} - S(T)re^{-rT}$$

$$= f(T)e^{-rT} + S'(T)e^{-rT} - S(T)re^{-rT} = e^{-rT}[f(T) + S'(T) - rS(T)].$$

(To differentiate the integral with respect to T, use formula (6) on p. 318 in EMEA, or formula (9) on p. 337 in MA I.) Obviously,

$$F'(T) = 0 \iff f(T) + S'(T) - rS(T) = 0 \iff f(T) = rS(T) - S'(T).$$

Notes on grading: For your guidance, we have indicated an approximate grade on each paper. On the 74 papers handed in, the distribution of grades turned out as follows:

A: 24, B: 22, C: 12, D: 8, E: 8.

(I believe that adds up to 74.) If you got a D you should take that as a very serious warning, and any paper rewarded with an E would most likely fail at a real exam.

But remember that the grades given for this paper will *not* count towards your final grade for this course.

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