Department of Economics
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ECON3120/4120 Mathematics 2, spring 2004
Problem solutions for seminar no. 1, 26–30 January 2004

(For practical reasons some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 3.4.5 (= MA I, 2.7.5)

(a)
$$\begin{aligned} x+2=\sqrt{4x+13} &\overset{*}{\implies} (x+2)^2 = (\sqrt{4x+13}\,)^2 \\ &\iff x^2+4x+4 = 4x+13 \\ &\iff x^2 = 9 \iff x = \pm 3. \end{aligned}$$

This shows that *if* x satisfies $x+2=\sqrt{4x+13}$, *then* x must equal 3 or -3, but *it does not show* that these values of x actually satisfy the equation. The problem is that the implication $\overset{*}{\implies}$ only runs from the left towards the right — we cannot draw any conclusions in the opposite direction. (If $a^2=b^2$, we know only that $a=b$ or $a=-b$.) This means that "extra" solutions may sneak in, in the sense that there may be solutions of the final equation that are not solutions of the original equation.

Checking the solutions, we find that $x=3$ is indeed a solution of the given equation, whereas $x=-3$ is not. For $x=-3$ the left-hand side (LHS) becomes $x+2=-3+2=-1$, and the right-hand side (RHS) becomes $\sqrt{4x+13} = \sqrt{-12+13} = \sqrt{1} = 1$. Thus LHS $\neq$ RHS, and $x=-3$ is not a solution of the equation. Remember, $\sqrt{1}=1$, not -1.

It follows that the equation $x+2=\sqrt{4x+13}$ has only one solution, $x=3$.

(b) Squaring both sides of the equation $|x+2|=\sqrt{4-x}$ yields

$$\begin{aligned} |x+2|^2 = (\sqrt{4-x}\,)^2 &\iff (x+2)^2 = 4-x \iff x^2+4x+4 = 4-x \\ &\iff x^2+5x=0 \iff x(x+5)=0. \end{aligned}$$

The last equation has the solutions $x=0$ and $x=-5$. (A product equals 0 precisely when at least one factor equals 0.) Checking the solutions, we easily find that both $x=0$ and $x=-5$ are solutions of the original equation as well.

(c) If $x\geq 0$, then $|x|=x$, and we get the quadratic equation $x^2-2x-3=0$, with the roots $x_1=3$ and $x_2=-1$. Since have assumed that $x\geq 0$, only $x_1=3$ can be used in this case.

If $x<0$, then $|x|=-x$, and $x^2+2x-3=0$, which has the roots $x_3=1$ and $x_4=-3$. But now we have assumed that $x<0$, so x_4 is the only usable solution in this case.

Conclusion: The equation $x^2-2|x|-3=0$ has the roots $x_1=3$ and $x_4=-3$ (and no others).

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(Alternative solution: Since $x^2 = |x|^2$, we must have $|x|^2 - 2|x| - 3 = 0$, which gives $|x| = 3$ or $|x| = -1$. But $|x| = -1$ is impossible, since $|x| \geq 0$. Hence $|x| = 3$, etc.)

EMEA, 3.4.6 (= MA I, 2.7.6)

(a) If x satisfies the equation

$$\sqrt{x-4} = \sqrt{x+5} - 9, \tag{1}$$

then x also satisfies the equation

$$x - 4 = (\sqrt{x+5} - 9)^2, \tag{2}$$

which we get by squaring both sides in (1). Calculating the square on the right-hand side of (2) gives

$$\begin{aligned} x - 4 = x + 5 - 18\sqrt{x+5} + 81 &\iff 18\sqrt{x+5} = 90 \\ &\iff \sqrt{x+5} = 5 \\ &\iff x + 5 = 25 \iff x = 20. \end{aligned}$$

(The last four transitions really are equivalences, but only $\Longrightarrow$ is really needed here.) This shows that *if* x is a solution of (1), then $x = 20$. No other value of x can satisfy (1). *But if we check this solution*, we find that with $x = 20$ the LHS of (1) becomes $\sqrt{16} = 4$, and the RHS becomes $\sqrt{25} - 9 = 5 - 9 = -4$. Thus the LHS and the RHS are different. This means that equation (1) actually has no solutions at all.

(b) If x is a solution of

$$\sqrt{x-4} = 9 - \sqrt{x+5}, \tag{3}$$

then just as in part (a) we find that x must be a solution of

$$x - 4 = (9 - \sqrt{x+5}\,)^2. \tag{4}$$

Now, $(9 - \sqrt{x+5}\,)^2 = (\sqrt{x+5} - 9)^2$, so equation (4) is equivalent to equation (2) in part (a). This means that (4) has exactly one solution, namely $x = 20$. Inserting this value of x into equation (3), we find that $x = 20$ *is* a solution of (3).

We see that the two solid curves in the figure have no point in common, that is, the expressions $\sqrt{x-4}$ and $\sqrt{x+5} - 9$ are not equal for any value of x. (Taking the derivative, we can easily see that the difference $\sqrt{x-4} - (\sqrt{x+5} - 9)$ increases with x, so there is no point of intersection farther to the right, either.) This explains why the equation in (a) has no solution. The dashed curve $y = 9 - \sqrt{x+5}$, on the other hand, intersects $y = \sqrt{x+5}$ for $x = 20$ (and only there), and this corresponds to the solution in part (b).

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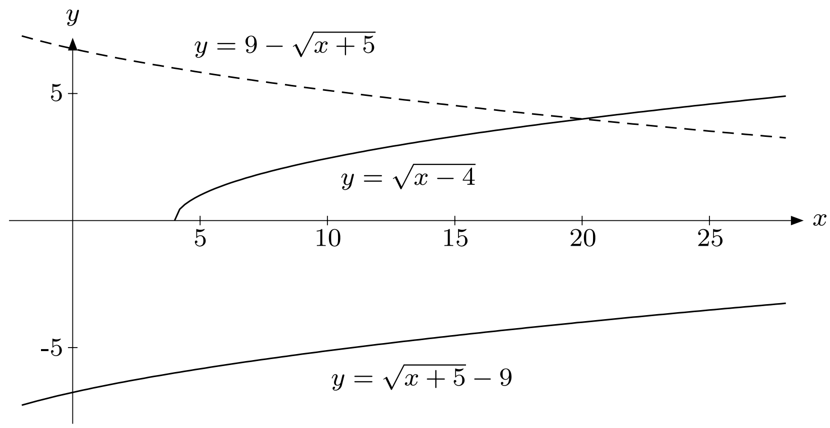

Problem 3.4.6 (2.7.6 in MA I)

Comment: In part (a) it was necessary to check the result, because the transition from (1) to (2) is only an implication, not an equivalence. Similarly, it was necessary to check the result in part (b), since the transition from (3) to (4) also is only an implication — at least, it is not clear that it is an equivalence. (Afterwards, it turned out to be an equivalence, but we could not know that until we had solved the equation.)

EMEA, 3.4.7 ((a),(b),(c) = MA I, 2.7.7(a),(e),(c))

See the answers in the back of the book.

(b) (= (e) in MA I) It is easy to see by means of a sign diagram that $x(x+3) < 0$ precisely when x lies in the open interval $(-3, 0)$. Therefore we have an implication from left to right (that is, "only if"), but not in the other direction. There are in fact values of x greater that -3 for which $x(x+3)$ is not negative, $x = 10$, for example.

(c) If $x = -5$, for instance, we have $x < 3$ but $x^2 > 9$. Hence we cannot have "if" here.

EMEA, 3.4.8 (= MA I, 2.7.8)

(a) The given text contains the following implications:

$$\begin{aligned} x + \sqrt{x+4} = 2 \;&\Longrightarrow \sqrt{x+4} = 2 - x \\ &\Longrightarrow x + 4 = 4 - 4x + x^2 \\ &\Longrightarrow\; x^2 - 5x = 0 \\ &\Longrightarrow\; x - 5 = 0 \\ &\Longleftarrow\; x = 5 \end{aligned}$$

The implications in line 1–3 are true (although the ones in line 1 and line 3 can be strengthened to equivalences.) But the implication in line 4 is false (we could have $x = 0$). Instead there should be an implication from right to left. The implication

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in the final line is correct, but it is the opposite implication we need. (That one is also true.) The way things stand, the argument is unusable as a solution of the equation. What we need is a chain of implications from left to right, that is, from the given equation to the answer. We also *must check the answer*, unless we have equivalences all the way. Remember that extra "solutions" may sneak in on the way when we only have implications.

The problem text in the Norwegian book is slightly different from the English one, so here is the Norwegian version of the answer to part (a):

Den gitte teksten inneholder følgende implikasjoner:

$$\begin{aligned} x + \sqrt{x+4} = 2 \quad &\Longrightarrow \sqrt{x+4} = 2 - x \\ &\Longleftrightarrow x + 4 = 4 - 4x + x^2 \\ &\Longrightarrow x^2 - 5x = 0 \\ &\Longrightarrow x - 5 = 0 \\ &\Longleftarrow x = 5 \end{aligned}$$

Implikasjonen i første linje er korrekt (men kan forsterkes til en ekvivalens.) Ordet "omformes" må bety at det dreier seg om en ekvivalens i 2. linje, men i virkeligheten har vi bare en implikasjon her (mot høyre). Implikasjonen i 3. linje er riktig (selv om den kan forsterkes til en ekvivalens). Derimot er implikasjonen i 4. linje gal (vi kunne jo ha $x = 0$). I stedet skulle det her være en implikasjon mot venstre. Implikasjonen i siste linje er korrekt, men det er den motsatte implikasjonen vi trenger. (Den er også riktig.) Slik som det hele står, er det ubrukelig som løsning av ligningen. Vi trenger jo en kjede av implikasjoner som alle går mot høyre, dvs. fra forutsetningen mot konklusjonen. Dessuten *må vi sette prøve på svaret*, hvis vi da ikke har ekvivalenser hele veien. Det kan jo snike seg inn ekstra løsninger underveis når vi bare har implikasjoner.

End of Norwegian text.

(b) A correct solution of the equation could be as follows: "We have

$$\begin{aligned} x + \sqrt{x+4} = 2 \quad &\Longleftrightarrow \quad \sqrt{x+4} = 2 - x \\ &\Longrightarrow x + 4 = 4 - 4x + x^2 \\ &\Longleftrightarrow x^2 - 5x = 0 \Longleftrightarrow x(x-5) = 0 \\ &\Longleftrightarrow x = 0 \text{ or } x = 5 \end{aligned}$$

Thus the only values of x that can possibly satisfy the equation are $x = 0$ and $x = 5$. Checking the answers, we find that $x = 0$ is a solution, whereas $x = 5$ is not."

In the formulas here there are equivalence arrows all the way, except in line 2. But precisely because of this exception we do not have a chain of implications from the statement "$x = 0$ or $x = 5$" back to the given equation. Therefore we must check the answers. Of course, it was the squaring operation (from line 1 to line 2) that allowed the extraneous solution to creep in.

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EMEA, 4.10.1 (= MA I, 3.10.2)

We use the rule $\ln x^p = p \ln x$ and get

(a) $\ln 9 = \ln 3^2 = 2 \ln 3$,

(b) $\ln \sqrt{3} = \ln 3^{1/2} = \dfrac{1}{2} \ln 3$,

(c) $\ln \sqrt[5]{3^2} = \ln 3^{2/5} = \dfrac{2}{5} \ln 3$,

(d) $\ln \dfrac{1}{81} = \ln \dfrac{1}{3^4} = \ln 3^{-4} = -4 \ln 3$.

EMEA, 4.10.3 (= MA I, 3.10.4)

(a) $$3^x 4^{x+2} = 8 \iff 3^x 4^x 4^2 = 8 \iff 12^x = 8/4^2 = 1/2.$$

If we take (the natural) logarithm on both sides, we get

$$x \ln 12 = \ln 1/2 = -\ln 2,$$

so

$$x = -\ln 2 / \ln 12.$$

(b) We use $\ln x^2 = 2 \ln x$, and get

$$3 \ln x + 2 \ln x^2 = 6 \iff 7 \ln x = 6 \iff \ln x = 6/7 \iff x = e^{6/7}.$$

(c) $$\begin{aligned} 4^x - 4^{x-1} = 3^{x+1} - 3^x &\iff 4^x(1 - 4^{-1}) = 3^x(3 - 1) \\ &\iff \frac{4^x}{3^x} = \frac{2}{1 - \frac{1}{4}} = \frac{8}{3} \\ &\iff (4/3)^x = 8/3 \iff x \ln(4/3) = \ln(8/3) \\ &\iff x = \frac{\ln(8/3)}{\ln(4/3)}. \end{aligned}$$

EMEA, 6.10.1 (= MA I, 5.10.1)

In these problems we use the usual rules for derivatives of sums, products, etc., remembering all the time that $(d/dx)e^x = e^x$. See the answers in the back of the book.

EMEA, 6.11.3 (= MA I, 5.11.3)

For these problems we need the chain rule. That is an important rule! In particular, we need the fact that $\dfrac{d}{dx} \ln f(x) = \dfrac{1}{f(x)} f'(x) = \dfrac{f'(x)}{f(x)}$ when f is a differentiable function (with $f(x) > 0$).

(a) $y = \ln(\ln x) = \ln u \implies y' = \dfrac{1}{u}u' = \dfrac{1}{\ln x}\dfrac{1}{x} = \dfrac{1}{x \ln x}$.

(b) $y = \ln\sqrt{1-x^2} = \ln u \implies y' = \dfrac{1}{u}u' = \dfrac{1}{\sqrt{1-x^2}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{-x}{1-x^2}$.
(Alternatively: $\sqrt{1-x^2} = (1-x^2)^{1/2} \implies y = \frac{1}{2}\ln(1-x^2)$, and so on.)

(c) $y = e^x \ln x \implies y' = e^x \ln x + e^x \dfrac{1}{x} = e^x\left(\ln x + \dfrac{1}{x}\right)$.

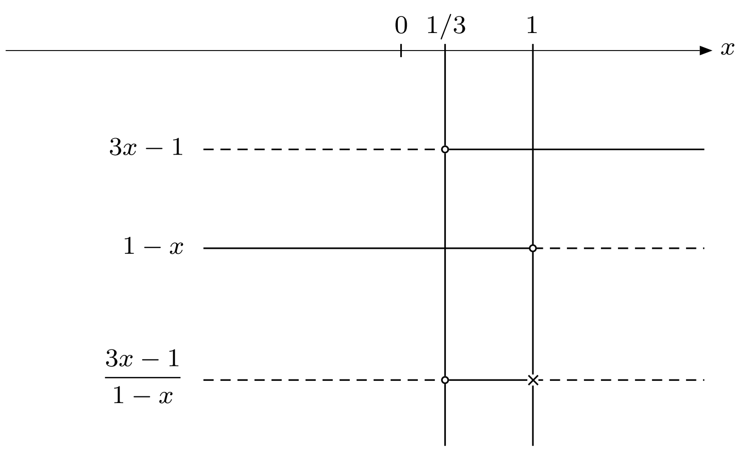
(d) $y = e^{x^3} \ln x^2 \implies y' = 3x^2 e^{x^3} \ln x^2 + e^{x^3}\dfrac{1}{x^2}2x = e^{x^3}\left(3x^2 \ln x^2 + \dfrac{2}{x}\right)$.

(e) $y = \ln(e^x + 1) \implies y' = \dfrac{e^x}{e^x + 1}$.

(f) $y = \ln(x^2 + 3x - 1) \implies y' = \dfrac{2x+3}{x^2+3x-1}$.

EMEA, 6.11.4 (= MA I, 5.11.4)

(b) We must have $1 - x \neq 0$ for the fraction to be defined, and $\dfrac{3x-1}{1-x} > 0$ for the logarithm to be defined.



Problem EMEA 6.11.4(b)

The sign diagram shows that $\dfrac{3x-1}{1-x}$ is defined and positive if and only if $1/3 < x < 1$.

(c) $\ln|x|$ is defined $\iff |x| > 0 \iff x \neq 0$.

(f) The fraction $\dfrac{1}{\ln(\ln x) - 1}$ is defined when $\ln(\ln x)$ is defined and different from 1. It is clear that $\ln(\ln x)$ is defined when $\ln x$ is defined and positive, that is, for $x > 1$. Further, $\ln(\ln x) = 1 \iff \ln x = e \iff x = e^e$. Conclusion:

$$\frac{1}{\ln(\ln x) - 1} \text{ is defined } \iff x > 1 \text{ and } x \neq e^e.$$

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EMEA, 6.11.2 (= MA I, 5.11.2)

(a) $$\frac{dy}{dx} = 3x^2(\ln x)^2 + x^3 2\ln x \cdot \frac{d}{dx}(\ln x) = 3x^2(\ln x)^2 + 2x^3\ln x \cdot \frac{1}{x}$$
$$= x^2\ln x(3\ln x + 2).$$

(b) $$\frac{dy}{dx} = \frac{2x\ln x - x^2(1/x)}{(\ln x)^2} = \frac{x(2\ln x - 1)}{(\ln x)^2}\,.$$

(c) The chain rule gives $\dfrac{dy}{dx} = 10(\ln x)^9\dfrac{d}{dx}(\ln x) = \dfrac{10(\ln x)^9}{x}\,.$

(d) The chain rule here too:

$$\frac{dy}{dx} = 2(\ln x + 3x)\frac{d}{dx}(\ln x + 3x) = 2(\ln x + 3x)(\frac{1}{x} + 3).$$

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