Department of Economics
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ECON3120/4120 Mathematics 2, spring 2004

Problem solutions for seminar no. 2, 2–6 February 2004

(For practical reasons some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 6.11.10 (= MA I, 5.11.8)

In these problems we can use logarithmic differentiation. Alternatively we can write the functions in the form $f(x) = e^{g(x)}$ and then use the fact that

$$f'(x) = e^{g(x)}g'(x) = f(x)g'(x).$$

(a) Let $f(x) = (2x)^x$. Then $\ln f(x) = x\ln(2x)$, so

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(x\ln(2x)) = 1 \cdot \ln(2x) + x \cdot \frac{1}{2x} \cdot 2 = \ln(2x) + 1.$$

Hence,

$$f'(x) = f(x)(\ln(2x) + 1) = (2x)^x(\ln x + \ln 2 + 1).$$

Alternatively: $f(x) = e^{\ln f(x)} = e^{x\ln(2x)}$ yields

$$f'(x) = e^{x\ln(2x)}\frac{d}{dx}(x\ln(2x)) = (2x)^x(\ln(2x) + 1) = (2x)^x(\ln x + \ln 2 + 1).$$

(b) We have $f(x) = x^{\sqrt{x}} = \left(e^{\ln x}\right)^{\sqrt{x}} = e^{\sqrt{x}\ln x}$, so

$$\begin{aligned} f'(x) = \frac{d}{dx}e^{\sqrt{x}\ln x} &= e^{\sqrt{x}\ln x} \cdot \frac{d}{dx}(\sqrt{x}\ln x) \\ &= x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right). \end{aligned}$$

With logarithmic differentiation we do as follows: $\ln f(x) = \sqrt{x}\ln x$, and therefore

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln f(x) = \left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right), \quad \text{etc.}$$

(c) With $f(x) = \left(\sqrt{x}\right)^x$ we get $\ln f(x) = x\ln\sqrt{x} = \frac{1}{2}x\ln x$, so

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\frac{d}{dx}(x\ln x) = \frac{\ln x + 1}{2}$$

which gives $f'(x) = f(x)\frac{1}{2}(\ln x + 1) = \frac{1}{2}(\sqrt{x})^x(\ln x + 1)$.

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(d) If we let $g(x) = x^{(x^x)}$, we get $\ln g(x) = x^x \ln x$. To find the derivative of this function, we need the derivative of x^x. A simple calculation gives

$$\frac{d}{dx}x^x = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\cdot\frac{d}{dx}(x\ln x) = x^x(\ln x + 1).$$

Using logarithmic differentiation, we then find

$$\frac{g'(x)}{g(x)} = \frac{d}{dx}(x^x\ln x) = \big(x^x(\ln x+1)\big)\cdot\ln x + x^x\cdot\frac{1}{x} = x^x\big((\ln x)^2 + \ln x + \frac{1}{x}\big)$$

and consequently

$$g'(x) = x^{(x^x)}\cdot x^x\big((\ln x)^2+\ln x+\frac{1}{x}\big) = x^{(x^x+x)}\big((\ln x)^2+\ln x+\frac{1}{x}\big).$$

EMEA, 7.9.1 (= MA I, 6.6.1)

(a) Let $f(x) = x^7 - 5x^5 + x^3 - 1$. Then f is continuous everywhere, and in particular in the closed interval $[-1,1]$. Since $f(-1) = 2 > 0$ and $f(1) = -4 < 0$, there must be a point c in $(-1,1)$ with $f(c) = 0$. See the intermediate value theorem, Theorem 7.9.1 on p. 255 ("Skjæringssetningen", Theorem 6.6.1 on p. 226 in MA I).

(b) In the same way as part (a). We let $f(x) = x^3 + 3x - 8$ and notice that $f(-2) = -22 < 0$ and $f(3) = 28 > 0$.

(c) Here we let $f(x) = \sqrt{x^2+1} - 3x$ and find that $f(0) = 1 > 0$, whereas $f(1) = \sqrt{2} - 3 < 0$.

(d) The function $f(x) = e^{x-1} - 2x$ is continuous everywhere and $f(0) = e^{-1} > 0$, $f(1) = e^0 - 2 = -1 < 0$, so the intermediate value theorem ensures that there is a zero of f in $(0,1)$.

EMEA, 10.2.2 (= MA I, 8.2.2)

(a) (i) $1000\cdot 1.05^{10} = 1628.8946 \approx 1629$

(ii) $1000\cdot\left(1+\dfrac{0.05}{12}\right)^{12\cdot 10} = 1647.0095 \approx 1647$

(iii) $1000\cdot e^{0.05\cdot 10} = 1000e^{0.5} = 1648.7213 \approx 1649$

(b) (i) $1000\cdot 1.05^{50} = 11467.400 \approx 11467$

(ii) $1000\cdot\left(1+\dfrac{0.05}{12}\right)^{12\cdot 50} = 12119.383 \approx 12119$

(iii) $1000\cdot e^{0.05\cdot 50} = 1000e^{2.5} = 12182.494 \approx 12182$

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Exam problem 22

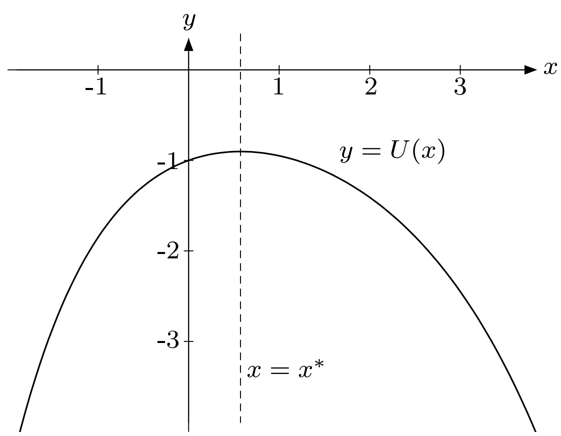
(a) $U'(x) = aAe^{-ax} - bBe^{bx}$, $U''(x) = -a^2 Ae^{-ax} - b^2 Be^{bx}$.

The function U is differentiable everywhere, so any extreme point must be a stationary point.

$$
\begin{aligned}
U'(x) = 0 &\iff bBe^{bx} = aAe^{-ax} \iff \ln(bBe^{bx}) = \ln(aAe^{-ax}) \\
&\iff \ln(bB) + bx = \ln(aA) - ax \\
&\iff (a+b)x = \ln(aA) - \ln(bB) = \ln\Big(\frac{aA}{bB}\Big) \\
&\iff x = \frac{1}{a+b}\ln\Big(\frac{aA}{bB}\Big) = x^*.
\end{aligned}
$$

Hence x^* is the only stationary point of U. Moreover, $U''(x) < 0$ for all x, so $U'(x)$ is strictly decreasing everywhere. It follows that $U'(x) > 0$ for $x < x^*$ and $U'(x) < 0$ for $x > x^*$. By the first-derivative test, x^* is a (global) maximum point for U. (See p. 273 in EMEA, p. 292 in MA I.)

(b) It was shown in (a) that $U''(x) < 0$ for all x. Hence U is concave everywhere. The diagram shows the graph of U together with the straight line $x = x^*$ when $A = 0.6$, $B = 0.4$, $a = 1$, $b = 0.4$, and $x^* = (\ln 2.5)/1.6 \approx 0.5727$.



Exam problem 22

(c) The standard rules for powers yield

$$U(x) = -Ae^{-ax} - Be^{bx} = -Ae^{-ax^*}e^{-a(x-x^*)} - Be^{bx^*}e^{b(x-x^*)}.$$

It remains to find a C such that

$$(1)\quad Ae^{-ax^*} = C/a \quad \text{and} \quad (2)\quad Be^{bx^*} = C/b.$$

Equation (1) gives $C = aAe^{-ax^*}$. We know from part (a) that $U'(x^*) = 0$, and so $aAe^{-ax^*} = bBe^{bx^*}$. Hence $C/b = aAe^{-ax^*}/b = Be^{bx^*}$, i.e. equation (2) is also satisfied.

The graph of U is symmetric about the vertical line $x = x^*$ if and only if $U(x^* + t) = U(x^* - t)$ for all t. From the formula we have just shown, it follows that if $a = b$, then

$$U(x^* + t) = -\frac{C}{a}e^{-at} - \frac{C}{b}e^{bt} = -\frac{C}{a}(e^{-at} + e^{at}),$$

and so $U(x^* + t) = U(x^* + (-t)) = U(x^* - t)$.

Exam problem 32

(a)
$$f'(x) = \frac{(xe^{2x})' \cdot (x+1) - xe^{2x} \cdot (x+1)'}{(x+1)^2} = \frac{e^{2x}(1+2x)(x+1) - xe^{2x}}{(x+1)^2} = \frac{e^{2x}(2x^2+2x+1)}{(x+1)^2}.$$

The domain of f is $D_f = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)$. The function is differentiable throughout its domain, so any local extreme points must be stationary points of f. The equation $2x^2 + 2x + 1 = 0$ has no real roots, and therefore f has no local extreme points. (If we try the formula for solving quadratic equations, we get $x = \dfrac{-1 \pm \sqrt{4-8}}{4}$.)

We also have $2x^2 + 2x + 1 = x^2 + (x+1)^2 > 0$ for all x, so $f'(x) > 0$ for all $x \neq -1$. This shows that f is strictly increasing in each of the intervals $(-\infty, -1)$ and $(-1, \infty)$.

(b) It is clear that $\lim_{x\to-1} xe^{2x} = -e^{-2} < 0$. When investigating the right-hand limit $\lim_{x\to(-1)^+} f(x)$, we need to determine what happens when x is close to but greater than -1. In particular, $x+1$ will then be positive and close to 0. It follows that

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} \frac{xe^{2x}}{x+1} = -\infty.$$

In a similar fashion we find that

$$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-} \frac{xe^{2x}}{x+1} = \infty,$$

since $x + 1$ is negative all the time as x tends to -1 from the left.

Further,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{x}{x+1} \cdot e^{2x}\right) = 1 \cdot 0 = 0$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{x}{x+1} \cdot e^{2x}\right) = \infty,$$

since $\lim_{x\to-\infty} x/(x+1) = \lim_{x\to\infty} x/(x+1) = 1$.

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(c) The second derivative of f is

$$f''(x) = \frac{d}{dx}\left(\frac{e^{2x}(2x^2+2x+1)}{(x+1)^2}\right)$$

$$= \frac{\left[2e^{2x}(2x^2+2x+1)+e^{2x}(4x+2)\right](x+1)^2 - e^{2x}(2x^2+2x+1)2(x+1)}{(x+1)^4}$$

$$= \cdots = \frac{e^{2x}(4x^3+8x^2+8x+2)}{(x+1)^3} = \frac{e^{2x}}{(x+1)^3}g(x),$$

where $g(x) = 4x^3+8x^2+8x+2$. Then $f''(x) = 0 \iff g(x) = 0$.

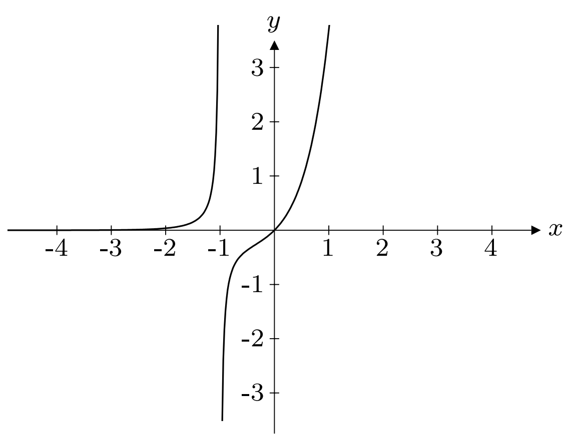
Since $g(-\frac{1}{2}) = -\frac{1}{2} < 0$ and $g(0) = 2 > 0$, there is a point x_0 in $(-\frac{1}{2}, 0)$ such that $g(x_0) = 0$. Moreover, $g'(x) = 12x^2+16x+8 = 4x^2+8(x+1)^2 > 0$ for all x. This shows that g is strictly increasing over the entire real line, and therefore $g(x) < 0$ for $x < x_0$ and $g(x) > 0$ for $x > x_0$. Hence x_0 is the only zero of g.

Since $f''(x)$ changes sign around $x = x_0$, x_0 must be an inflection point of f, and since x_0 is the only zero of f'', there are no other inflection points.

(d) The function f is convex in intervals where $f'' \geq 0$, and concave in intervals where $f'' \leq 0$. We know that $-1 < x_0$ and that

$$(x+1)^3 \begin{cases} < 0 & \text{if } x < -1, \\ > 0 & \text{if } x > -1, \end{cases} \qquad g(x) \begin{cases} < 0 & \text{if } x < x_0, \\ > 0 & \text{if } x > x_0. \end{cases}$$

A sign diagram for $f''(x) = e^{2x}g(x)/(x+1)^3$ then shows that f is convex over $(-\infty, -1)$, concave over $(-1, x_0]$ and convex again over $[x_0, \infty)$.



Exam problem 32 (d). The graph of $f(x) = \dfrac{xe^{2x}}{x+1}$.