**Department of Economics**
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# ECON3120/4120 Mathematics 2, spring 2004

## Problem solutions for seminar no. 3, 9–13 February 2004

(For practical reasons some of the solutions may include problem parts that were not on the problem list for the seminar.)

### EMEA, 5.3.2 (= MA I, 4.3.2)

We have

$$D = f(p) = \frac{157.8}{p^{0.3}} \iff p^{0.3} = \frac{157.8}{D} \iff p = g(D) = \Big(\frac{157.8}{D}\Big)^{1/0.3},$$

where $g$ then is the inverse function of $f$.

### EMEA, 5.3.9 (= MA I, 4.4.1)

(a) Let $g = f^{-1}$. Then

$$\begin{aligned} g(y) = x &\iff f(x) = y \iff (x^3-1)^{1/3} = y \iff x^3 - 1 = y^3 \\ &\iff x^3 = y^3 + 1 \iff x = (y^3+1)^{1/3}, \end{aligned}$$

so $g(y) = (y^3+1)^{1/3}$. If we use $x$ as the independent variable in $g$, we get $g(x) = (x^3+1)^{1/3}$. Here $D_g = V_f = \mathbb{R}$ and $V_g = D_f = \mathbb{R}$.

(b) We proceed as in part (a), but note that here the domain of $f$ is $D_f = (-\infty, 2) \cup (2, \infty) = \mathbb{R} \setminus \{2\}$. Hence (with $x \neq 2$),

$$\begin{aligned} g(y) = x &\iff f(x) = y \iff \frac{x+1}{x-2} = y \iff x + 1 = y(x-2) \\ &\iff (1-y)x = -2y - 1 \iff x = \frac{-2y-1}{1-y} = \frac{2y+1}{y-1}, \end{aligned}$$

and so $g(y) = (2y+1)/(y-1)$. Using $x$ as the independent variable, $g(x) = (2x+1)/(x-1)$. The domain of $g$ is $D_g = R_f = \mathbb{R} \setminus \{1\} = (-\infty, 1) \cup (1, \infty)$ and the range of $g$ is $R_g = D_f = \mathbb{R} \setminus \{2\}$.

Was there any danger of getting zero in the denominator of one of the fractions here? No, with $x = 2$ we would get the impossible equation $x + 1 = y(x-2) = 0$, contradicting $x = 2$. In the same way we cannot have $y = 1$ in the equation $(1-y)x = -2y - 1$.

(c) Here

$$\begin{aligned} f^{-1}(y) = x \iff & y = f(x) = (1 - x^3)^{1/5} + 2 \iff y - 2 = (1 - x^3)^{1/5} \\ \iff & (y-2)^5 = 1 - x^3 \iff x^3 = 1 - (y-2)^5 \\ \iff & x = (1 - (y-2)^5)^{1/3}, \end{aligned}$$

so $f^{-1}(y) = (1-(y-2)^5)^{1/3}$. With $x$ as the free variable, $f^{-1}(x) = (1-(x-2)^5)^{1/3}$.

In this problem $f$ and $f^{-1}$ are defined over all of $\mathbb{R}$, and $\mathbb{R}$ is also the range of both of them.

(Engelsk "range" og "domain" = norsk "verdimengde" og "definisjonsmengde".)

**EMEA, 7.11.2(a) (= MA I, 6.5.1(b))**

L'Hôpital's rule yields

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \text{“}\frac{0}{0}\text{”} = \lim_{x \to a} \frac{2x}{1} = 2a.$$

But note that we don't really need l'Hôpital's rule here, because

$$x^2 - a^2 = (x+a)(x-a),$$

and therefore

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a.$$

**EMEA, 7.11.4 (= MA I, 6.5.3)**

The first fraction gives an indeterminate ("0/0") form as $x \to 1$, but the second fraction does not. Therefore we cannot use l'Hôpital's rule to evaluate the second limit. The correct answer is

$$\lim_{x \to 1} \frac{x^2 + 3x - 4}{2x^2 - 2x} = \text{“}\frac{0}{0}\text{”} = \lim_{x \to 1} \frac{2x + 3}{4x - 2} = \frac{5}{2}.$$



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**Exam problem 18**

We shall have to use l'Hôpital's rule twice:

$$\lim_{x\to 0}\frac{e^x-1-x}{x\sqrt{1+x}-x} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 0}\frac{e^x-1}{\sqrt{1+x}+\dfrac{x}{2\sqrt{1+x}}-1}$$

$$\overset{*}{=} \lim_{x\to 0}\frac{(e^x-1)2\sqrt{1+x}}{2(1+x)+x-2\sqrt{1+x}} = \lim_{x\to 0}2\sqrt{1+x}\cdot\lim_{x\to 0}\frac{e^x-1}{2+3x-2\sqrt{1+x}}$$

$$= 2\lim_{x\to 0}\frac{e^x-1}{2+3x-2\sqrt{1+x}} = \text{“}\frac{0}{0}\text{”} = 2\lim_{x\to 0}\frac{e^x}{3-\dfrac{1}{\sqrt{1+x}}} = 2\cdot\frac{1}{3-1} = 1.$$

At $\overset{*}{=}$ we rearrange the fraction (multiplying the numerator and the denominator by $2\sqrt{1+x}$), so that we shall not have to differentiate the fraction $x/2\sqrt{1+x}$. This transformation is not necessary, but simplifies the computation.

**Exam problem 22**

(a) $U'(x) = aAe^{-ax} - bBe^{bx}$, $U''(x) = -a^2Ae^{-ax} - b^2Be^{bx}$.

The function $U$ is differentiable everywhere, so any extreme point must be a stationary point.

$$\begin{aligned} U'(x) = 0 &\iff bBe^{bx} = aAe^{-ax} \iff \ln(bBe^{bx}) = \ln(aAe^{-ax}) \\ &\iff \ln(bB) + bx = \ln(aA) - ax \\ &\iff (a+b)x = \ln(aA) - \ln(bB) = \ln\Big(\frac{aA}{bB}\Big) \\ &\iff x = \frac{1}{a+b}\ln\Big(\frac{aA}{bB}\Big) = x^*. \end{aligned}$$

Hence $x^*$ is the only stationary point of $U$. Moreover, $U''(x) < 0$ for all $x$, so $U'(x)$ is strictly decreasing everywhere. It follows that $U'(x) > 0$ for $x < x^*$ and $U'(x) < 0$ for $x > x^*$. By the first-derivative test, $x^*$ is a (global) maximum point for $U$. (See p. 273 in EMEA, p. 292 in MA I.)

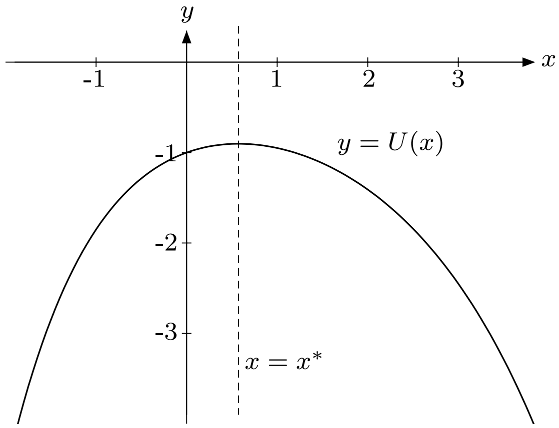
(b) It was shown in (a) that $U''(x) < 0$ for all $x$. Hence $U$ is concave everywhere. The diagram shows the graph of $U$ together with the straight line $x = x^*$ when $A = 0.6$, $B = 0.4$, $a = 1$, $b = 0.4$, and $x^* = (\ln 2.5)/1.6 \approx 0.5727$.

(c) The standard rules for powers yield

$$U(x) = -Ae^{-ax} - Be^{bx} = -Ae^{-ax^*}e^{-a(x-x^*)} - Be^{bx^*}e^{b(x-x^*)}.$$

It remains to find a $C$ such that

$$(1)\ \ Ae^{-ax^*} = C/a \quad\text{and}\quad (2)\ \ Be^{bx^*} = C/b.$$




Exam problem 22

Equation (1) gives $C = aAe^{-ax^*}$. We know from part (a) that $U'(x^*) = 0$, and so $aAe^{-ax^*} = bBe^{bx^*}$. Hence $C/b = aAe^{-ax^*}/b = Be^{bx^*}$, i.e. equation (2) is also satisfied.

The graph of $U$ is symmetric about the vertical line $x = x^*$ if and only if $U(x^* + t) = U(x^* - t)$ for all $t$. From the formula we have just shown, it follows that if $a = b$, then

$$U(x^* + t) = -\frac{C}{a}e^{-at} - \frac{C}{b}e^{bt} = -\frac{C}{a}(e^{-at} + e^{at}),$$

and so $U(x^* + t) = U(x^* + (-t)) = U(x^* - t)$.

(d) The formula for $U(x)$ in part (c) gives

$$U'(x) = Ce^{-a(x-x^*)} - Ce^{b(x-x^*)}$$
$$U''(x) = -Cae^{-a(x-x^*)} - Cbe^{b(x-x^*)}$$

Hence,

$$U(x^*) = -\frac{C}{a} - \frac{C}{b} = -C\Big(\frac{1}{a} + \frac{1}{b}\Big),$$
$$U'(x^*) = 0 \quad \text{(we already found that in (a),}$$
$$U''(x^*) = -C(a+b).$$

The quadratic approximation to $U(x)$ around $x^*$ is therefore

$$U(x^*) + U'(x^*)(x - x^*) + \frac{1}{2}U''(x^*)(x - x^*)^2 = -C\Big(\frac{1}{a} + \frac{1}{b}\Big) - \frac{1}{2}C(a+b)(x - x^*)^2.$$

**Exam problem 27**

Let $f(x) = \sqrt[3]{x+1} - \sqrt{x-3}$. Then $f(7) = \sqrt[3]{8} - \sqrt{4} = 2 - 2 = 0$, and our problem is to find

$$\lim_{x\to 7} \frac{f(x) - f(7)}{x - 7}\,.$$



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But this is simply $f'(7)$. (You did recognize the Newton quotient, didn't you?) Now,

$$f'(x) = \frac{1}{3}(x+1)^{-2/3} - \frac{1}{2}(x-3)^{-1/2} = \frac{1}{3(\sqrt[3]{x+1}\,)^2} - \frac{1}{2\sqrt{x-3}}\,,$$

and so

$$f'(7) = \frac{1}{3(\sqrt[3]{8}\,)^2} - \frac{1}{2\sqrt{4}} = \frac{1}{3\cdot 2^2} - \frac{1}{2\cdot 2} = -\frac{1}{6}\,.$$

Of course, we could have used l'Hôpital's rule:

$$\lim_{x\to 7}\frac{f(x)}{x-7} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 7}\frac{f'(x)}{1} = f'(7) = -\frac{1}{6}\,,$$

which would involve virtually the same operations.

**Exam problem 68**

(a) The derivative of $h$ is

$$h'(x) = \frac{e^x(2+e^{2x}) - e^x 2e^{2x}}{(2+e^{2x})^2} = \frac{e^x(2-e^{2x})}{(2+e^{2x})^2}\,.$$

The sign of $h'(x)$ is determined by the factor $2-e^{2x}$, and we find that

$$h'(x)\begin{cases} >0 & \text{for } x < \frac{1}{2}\ln 2 \\ =0 & \text{for } x = \frac{1}{2}\ln 2 \\ <0 & \text{for } x > \frac{1}{2}\ln 2 \end{cases}$$

Therefore $h$ is (strictly) increasing in $(-\infty, \frac{1}{2}\ln 2\,]$ and (strictly) decreasing in $[\,\frac{1}{2}\ln 2, \infty)$. Thus we see that $h$ has a global maximum point at $x = \frac{1}{2}\ln 2 = \ln\sqrt{2}$, but no other global or local extreme points.

(b) The function $h$ is strictly increasing in the interval $(-\infty, 0)$. Since $\lim\limits_{x\to-\infty} h(x) = 0$ and $\lim\limits_{x\to 0^-} h(x) = h(0) = 1/3$, the restriction of $h$ to $(-\infty, 0)$ has an inverse function $p : (0, 1/3) \to (-\infty, 0)$.

Then $y = p(x) \iff [h(y) = x$ and $y < 0]$. Furthermore,

$$h(y) = x \iff \frac{e^y}{2+e^{2y}} = x \iff (2+e^{2y})x = e^y \iff x\big(e^y\big)^2 - e^y + 2x = 0.$$

The last equation is a quadratic equation for $e^y$ with the roots

$$e^y = \frac{1 \pm \sqrt{1-8x^2}}{2x}\,.$$

Now, if $0 < x < 1/3$, then $2x < 2/3$, and

$$\frac{1 + \sqrt{1 - 8x^2}}{2x} > \frac{1}{2x} > \frac{3}{2} > 1.$$

Since we are looking for a $y < 0$, we must have $e^y < 1$, and so we get

$$y = p(x) = \ln\left(\frac{1 - \sqrt{1 - 8x^2}}{2x}\right) = \ln\left(1 - \sqrt{1 - 8x^2}\,\right) - \ln(2x).$$

(c) The function $f$ will be differentiable everywhere and

$$f'(x) = \frac{g'(x)\bigl(2 + (g(x))^2\bigr) - g(x)2g(x)g'(x)}{\bigl(2 + (g(x))^2\bigr)^2} = \frac{g'(x)\bigl(2 - (g(x))^2\bigr)}{\bigl(2 + (g(x))^2\bigr)^2}\,.$$

Since $g'(x) > 0$ for all $x$, the sign of $f'(x)$ is determined by the factor $\bigl(2 - (g(x))^2\bigr)$. Even though $g$ is strictly increasing, it is certainly possible to have $|g(x)| < \sqrt{2}$ for all $x$. (We could for instance have $g(x) = e^x/(1 + e^x)$.) If such is the case, then $f'(x) > 0$ for all $x$. On the other hand we could also have $|g(x)| > \sqrt{2}$ for all $x$, and then $f'(x) < 0$ everywhere.

Thus the function $f$ need not have any stationary point; in particular, it need not have a maximum point. It is also clear that for suitable choices of $g$ it possible to have both strictly increasing and strictly decreasing functions $f$.



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