Department of Economics
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ECON3120/4120 Mathematics 2, spring 2004

Problem solutions for seminar no. 4, 16–20 February 2004

(For practical reasons some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 7.4.3 (= MA I, 7.5.4)

With $f(x) = 5(\ln(1+x) - \sqrt{1+x}\,) = 5\ln(1+x) - 5(1+x)^{1/2}$ we get

$$f'(x) = 5(1+x)^{-1} - \frac{5}{2}(1+x)^{-1/2},$$
$$f''(x) = -5(1+x)^{-2} + \frac{5}{4}(1+x)^{-3/2},$$

so

$$f(0) = -5, \quad f'(0) = \frac{5}{2}, \quad f''(0) = -\frac{15}{4},$$

and the Taylor polynomial of degree 2 about $x = 0$ is

$$f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = -5 + \frac{5}{2}x - \frac{15}{8}x^2.$$

EMEA, 7.5.2 (= MA I, 7.6.3)

(a) The cube number (third power) closest to 25 is $3^3 = 27$. Thus we shall try to approximate $\sqrt[3]{25}$ by writing it as

$$\sqrt[3]{25} = \sqrt[3]{27-2} = 3\sqrt[3]{1-\tfrac{2}{27}} = 3(1-\tfrac{1}{27})^{1/3}.$$

The given approximation formula for $(1+x)^m$ gives

$$(1+x)^{1/3} \approx 1 + \tfrac{1}{3}x + \tfrac{1}{2}\tfrac{1}{3}(1-\tfrac{1}{3})x^2 = 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2.$$

Hence

$$\sqrt[3]{25} \approx 3\Big(1 + \frac{1}{3}\Big(-\frac{2}{27}\Big) - \frac{1}{9}\Big(\frac{2}{27}\Big)^2\Big) = 2\Big(1 - \frac{2}{81} - \frac{4}{6561}\Big) = \frac{6395}{2187} \approx 2.92409694.$$

A calculator gives $25^{1/3} \approx 2.92401774$.

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(b) Since $2^5 = 32$, we shall try the approximation

$$\sqrt[5]{33} = \sqrt[5]{32+1} = 2\sqrt[5]{1+\tfrac{1}{32}} = 2(1+\tfrac{1}{32})^{1/5}.$$

From the given approximation formula we get

$$(1+x)^{1/5} \approx 1 + \tfrac{1}{5}x - \tfrac{2}{25}x^2,$$

and so

$$\sqrt[5]{33} \approx 2\Big(1 + \frac{1}{5}\,\frac{1}{32} - \frac{2}{25}\,\frac{1}{1024}\Big) = 2\Big(1 + \frac{1}{160} - \frac{1}{12800}\Big) = \frac{12879}{6400} \approx 2.01234375.$$

A calculator gives $33^{1/5} \approx 2.01234662$.

EMEA, 9.1.1 (= MA I, 10.1.1)

This should be straightforward, since all the integrands are powers of x. Note that $x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$, $1/\sqrt{x} = x^{-1/2}$, and

$$\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{x^{3/2}}} = \sqrt{x \cdot x^{3/4}} = \sqrt{x^{7/4}} = x^{7/8}.$$

EMEA, 9.1.4 (= MA I, 10.1.2)

(a) $\displaystyle\int (t^3 + 2t - 3)\,dt = \int t^3\,dt + \int 2t\,dt - \int 3\,dt = \frac{t^4}{4} + t^2 - 3t + C.$

(b) $\displaystyle\int (x-1)^2\,dx = \int (x^2 - 2x + 1)\,dt = \frac{x^3}{3} - x^2 + x + C.$ Alternative solution: Since $\dfrac{d}{dx}(x-1)^3 = 3(x-1)^2$, we have $\displaystyle\int (x-1)^2\,dx = \frac{1}{3}(x-1)^3 + C_1$. This agrees with the first answer, because we have the relation $C_1 = C + 1/3$.

(c) $\displaystyle\int (x-1)(x+2)\,dx = \int (x^2 + x - 2)\,dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + C.$

(d) As in part (b), we can either first evaluate $(x+2)^3 = x^3 + 6x^2 + 12x + 8$, to get

$$\int (x+2)^3\,dx = \frac{x^4}{4} + 2x^3 + 6x^2 + 8x + C,$$

or calculate as follows:

$$\int (x+2)^3 = \frac{1}{4}(x+2)^4 + C_1.$$

Both answers are correct (with $C_1 = C - 4$).

(e) $\displaystyle\int (e^{3x} - e^{2x} + e^x)\,dx = \frac{e^{3x}}{3} - \frac{e^{2x}}{2} + e^x + C.$

(f) $\displaystyle\int \frac{x^3 - 3x + 4}{x}\,dx = \int \Big(x^2 - 3 + \frac{4}{x}\Big)dx = \frac{x^3}{3} - 3x + 4\ln|x| + C.$

EMEA, 9.1.5 (= MA I, 10.1.3)

(a) First simplify the integrand:

$$\frac{(y-2)^2}{\sqrt{y}} = \frac{y^2 - 4y + 4}{\sqrt{y}} = y^{3/2} - 4y^{1/2} + 4y^{-1/2}.$$

From this we get

$$\begin{aligned}\int \frac{(y-2)^2}{\sqrt{y}}\,dy &= \int y^{3/2} - 4y^{1/2} + 4y^{-1/2}\,dy \\ &= \frac{2}{5}y^{5/2} - \frac{8}{3}y^{3/2} + 8y^{1/2} + C = \frac{2}{5}y^2\sqrt{y} - \frac{8}{3}y\sqrt{y} + 8\sqrt{y} + C.\end{aligned}$$

(b) Polynomial division gives

$$\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}\,,$$

so

$$\int \frac{x^3}{x+1}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C.$$

(c) Since $\frac{d}{dx}(1+x^2)^{16} = 16(1+x^2)^{15}\cdot 2x = 32x(1+x^2)^{15}$,

$$\int x(1+x^2)^{15}\,dx = \frac{1}{32}(1+x^2)^{16}.$$

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Exam problem 1

(a) A simple sign diagram shows that $f(x) > 0$ when $0 < x < 1$ and when $x > 2$.

(b) $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$, hence $f'(x) = 3x^2 - 6x + 2$. The zeros of $f'(x)$ are the roots of the quadratic equation $3x^2 - 6x + 2 = 0$, namely

$$x_0 = 1 - \frac{\sqrt{3}}{3}\,, \quad x_1 = 1 + \frac{\sqrt{3}}{3}\,.$$

It is easy to see that $f'(x) > 0 \iff x < x_0$ or $x > x_1$. Also, $f'(x) < 0 \iff x_0 < x < x_1$. Hence, f is (strictly) increasing in $(-\infty, x_0]$ and in $[x_1, \infty)$, and (strictly) decreasing in $[x_0, x_1]$. It follows that x_0 is a local maximum point and x_1 is a local minimum point. The corresponding values of f are $f(x_0) = \frac{2}{9}\sqrt{3}$ and $f(x_1) = -\frac{2}{9}\sqrt{3}$.

(c) See the graph in the answer section of the exam problem collection.

$$\int_0^1 f(x)\,dx = \int_0^1 (x^3 - 3x^2 + 2x)\,dx = \Big|_0^1 \Big(\frac{x^4}{4} - x^3 + x^2\Big) = \frac{1}{4} - 0 = \frac{1}{4}\,.$$

Exam problem 49

(a) The derivative of the right-hand side equals the integrand.

(b) Use the result from (a) with $a = 3$ and $b = 1/2$. This yields

$$\begin{aligned}\int_0^4 7x\sqrt{x^2+9}\,dx &= 7\int_0^4 x(x^2+3^2)^{1/2}\,dx = 7\,\Big|_0^4 \frac{1}{2\cdot\frac{3}{2}}(x^2+3^2)^{3/2} \\ &= \frac{7}{3}(25^{3/2} - 9^{3/2}) = \frac{7}{3}(125 - 27) = \frac{7}{3}\cdot 98 = \frac{686}{3}\,.\end{aligned}$$

Exam problem 117

Using l'Hôpital's rule twice, we get

$$\lim_{x\to 1} \frac{\ln x - x + 1}{(x-1)^2} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 1} \frac{1/x - 1}{2(x-1)} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 1} \frac{-1/x^2}{2} = -\frac{1}{2}\,.$$

Alternatively, we can simplify the second fraction and get

$$\lim_{x\to 1} \frac{\ln x - x + 1}{(x-1)^2} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 1} \frac{1/x - 1}{2(x-1)} = \lim_{x\to 1} \frac{1-x}{2x(x-1)} = \lim_{x\to 1} \frac{-1}{2x} = -\frac{1}{2}\,,$$

using l'Hôpital's rule only once.

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