Department of Economics
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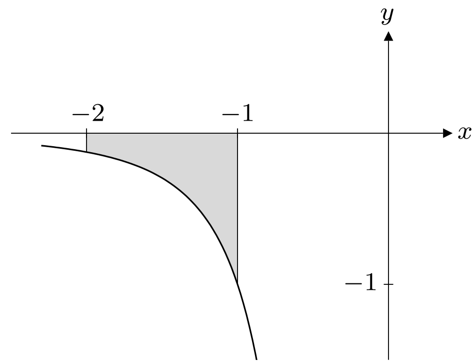
ECON3120/4120 Mathematics 2, spring 2004
Problem solutions for seminar no. 5, 23–27 February 2004

(For practical reasons some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 9.2.3 (= MA I, 10.2.3)



Let $F(x) = -1/2x^2 = -x^{-2}/2$. Then $F'(x) = x^{-3} = f(x)$. Since the graph of f lies below the x-axis throughout the interval in question, the area equals

$$-(F(-1) - F(-2)) = -\Big(-\frac{1}{2} + \frac{1}{8}\Big) = \frac{3}{8}\,.$$

With integral notation we can write this as

$$\text{Area} = -\int_{-2}^{-1} \frac{1}{x^3}\,dx = -\int_{-2}^{-1} x^{-3}\,dx = \Big|_{-2}^{-1} \frac{x^{-2}}{2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}\,.$$

EMEA, 9.4.2 (= MA I, 10.4.2)

(a) Let n be the total number of individuals. The number of individuals with income in the interval $[b, 2b]$ is then

$$N = n\int_b^{2b} Br^{-2}\,dr = n\Big|_b^{2b} -Br^{-1} = -\frac{nB}{2b} + \frac{nB}{b} = \frac{nB}{2b}\,.$$

Their total income is

$$M = n\int_b^{2b} Br^{-2}r\,dr = n\int_b^{2b} Br^{-1}\,dr = n\Big|_b^{2b} B\ln r = nB\ln 2.$$

Hence the mean income is $m = M/N = 2b\ln 2$.

(b) Total demand is

$$x(p) = \int_b^{2b} nD(p,r)f(r)\,dr = \int_b^{2b} nAp^\gamma r^\delta Br^{-2}\,dr = \int_b^{2b} nABp^\gamma r^{\delta-2}\,dr$$
$$= \Big|_b^{2b} nABp^\gamma \frac{r^{\delta-1}}{\delta-1} = nABp^\gamma b^{\delta-1}\frac{2^{\delta-1}-1}{\delta-1}\,.$$

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EMEA 9.5.1 (= MA I, 10.6.1)

(a)
$$\int \underset{f}{\underset{\uparrow}{x}}\, \underset{g'}{\underset{\uparrow}{e^{-x}}}\; dx = \underset{f}{\underset{\uparrow}{x}}\,(\underset{g}{\underset{\uparrow}{-e^{-x}}}) - \int \underset{f'}{\underset{\uparrow}{1}} \cdot (\underset{g}{\underset{\uparrow}{-e^{-x}}})\; dx$$
$$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

(b)
$$\int 3xe^{4x}\,dx = 3x \cdot \frac{1}{4}e^{4x} - \int 3 \cdot \frac{1}{4}e^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$$

(c)
$$\begin{aligned}\int (1+x^2)e^{-x}\,dx &= (1+x^2)(-e^{-x}) - \int 2x(-e^{-x})\,dx \\ &= -(1+x^2)e^{-x} + 2\int xe^{-x}\,dx \\ &= -(1+x^2)e^{-x} - 2xe^{-x} - 2e^{-x} + C \qquad \text{(use (a)!)} \\ &= -(x^2+2x+3)e^{-x} + C\end{aligned}$$

(d)
$$\int \underset{g'}{\underset{\uparrow}{x}}\, \underset{f}{\underset{\uparrow}{\ln x}}\; dx = \underset{g}{\underset{\uparrow}{\frac{x^2}{2}}}\, \underset{f}{\underset{\uparrow}{\ln x}} - \int \underset{g}{\underset{\uparrow}{\frac{x^2}{2}}}\, \underset{f'}{\underset{\uparrow}{\frac{1}{x}}}\; dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

EMEA 9.6.2 (= MA I, 10.7.2)

(b) With $u = g(x) = x^3 + 2$ we get $du = g'(x)\,dx = 3x^2\,dx$ and

$$\int x^2 e^{x^3+2}\,dx = \int e^{g(x)}\frac{1}{3}g'(x)\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{e^{x^3+2}}{3} + C.$$

(c) As a first attempt we could use the substitution $u = g(x) = x+2$, which gives $du = dx$ og

$$\int \frac{\ln(x+2)}{2x+4}\,dx = \int \frac{\ln u}{2u}\,du,$$

which is not very much simpler than the original integral. But if we notice that $\dfrac{\ln u}{u} = \ln u \cdot \dfrac{1}{u} = \ln u \cdot \dfrac{d}{du}\ln u$, then we can see that $v = \ln u$ yields $dv = \frac{1}{u}\,du$ and

$$\int \frac{\ln u}{2u}\,du = \int \frac{1}{2}v\,dv = \frac{1}{4}v^2 + C = \frac{1}{4}(\ln u)^2 + C = \frac{1}{4}(\ln(x+2))^2 + C.$$

With a little experience we would have noticed straight away that

$$\frac{\ln(x+2)}{2x+4} = \frac{\ln(x+2)}{2(x+2)} = \frac{1}{2}\ln(x+2)\frac{d}{dx}\ln(x+2),$$

which points directly to the substitution $v = \ln(x+2)$.

EMEA 9.6.3 (= MA I, 10.7.3)

See the answers in the back of the book. Instead of the substitution used there for part (a), we can also use $v = 1 + x^2$, which gives $dv = 2x\,dx$ and

$$\int_0^1 x\sqrt{1+x^2}\,dx = \int_0^1 \sqrt{1+x^2}\cdot x\,dx = \int_1^2 \sqrt{v}\cdot\frac{1}{2}\,dv = \Big|_1^2 \frac{1}{3}v^{3/2} = \frac{2\sqrt{2}-1}{3}\,.$$

Notice that the limits of integration usually change when we switch to a different variable of integration. Since $v = 1 + x^2$, we see that $x = 0$ and $x = 1$ correspond to $v = 1$ and $v = 2$, respectively. (Similarly, under the substitution $u = \sqrt{1+x^2}$ used in the book, $x = 0$ and $x = 1$ correspond to $u = 1$ og $u = \sqrt{2}$.)

EMEA 9.6.4 (= MA I, 10.7.4)

We want to solve the equation

$$\int_3^x \frac{2t-2}{t^2-2t}\,dt = \ln\Big(\frac{2}{3}x - 1\Big). \qquad (*)$$

For the right-hand side to have meaning, we must have $\frac{2}{3}x > 1$, that is, $x > \frac{3}{2}$. We substitute a new variable in order to calculate the integral on the left-hand side: With $u = t^2 - 2t$, we get $du = (2t-2)\,dt$. For $t = 3$ and $t = x$, we get $u = 3$ and $u = x^2 - 2x$, respectively, so the integral becomes

$$\int_3^x \frac{2t-2}{t^2-2t}\,dt = \int_3^{x^2-2x} \frac{du}{u} = \Big|_3^{x^2-2x} \ln u = \ln\Big(\frac{x^2-2x}{3}\Big).$$

Equation $(*)$ then yields

$$\frac{x^2-2x}{3} = \frac{2}{3}x - 1 \iff x^2 - 2x = 2x - 3 \iff x^2 - 4x + 3 = 0. \qquad (**)$$

The roots of equation $(**)$ are $x_1 = 3$ and $x_2 = 1$. Here x_2 is unusable as a solution of the original equation, because we have seen that we must have $x > 3/2$. (Also, the integral on the left would then become

$$\int_3^1 \frac{2t-2}{t^2-2t}\,dt = \int_3^1 \Big(\frac{1}{t} + \frac{1}{t-2}\Big)\,dt,$$

which does not converge.)

Thus we are left with the solution $x = 3$, and it is easy to check that this is indeed a solution of $(*)$. (Both sides of the equation become equal to 0.)

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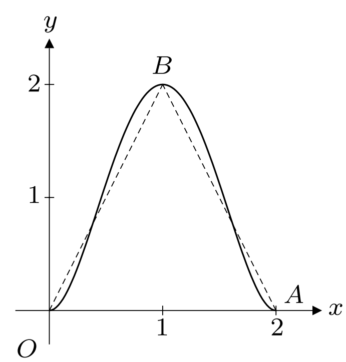
Exam problem 40

(a) $$\int (1-x^2)^2\,dx = \int (1-2x^2+x^4)\,dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$$

(b) $$\int_{P_N}^{P_L} (a - bP^{1-\alpha})\,dP = \Big|_{P_N}^{P_L} (aP - \frac{b}{2-\alpha}P^{2-\alpha})$$
$$= a(P_L - P_N) - \frac{b}{2-\alpha}(P_L^{2-\alpha} - P_N^{2-\alpha})$$

Exam problem 53

(a) $$\int_0^2 2x^2(2-x)^2\,dx = \int_0^2 (8x^2 - 8x^3 + 2x^4)\,dx$$
$$= \Big|_0^2 \Big(\frac{8}{3}x^3 - 2x^4 + \frac{2}{5}x^5\Big)$$
$$= \frac{8}{3}\cdot 8 - 2\cdot 16 + \frac{2}{5}\cdot 32$$
$$= \frac{32}{15} = 2\frac{2}{15}$$



We can see from the figure that the area between the graph and the x-axis over the interval $[0,2]$ must be approximately equal to the area of the triangle OAB, where the point O, A, and B are $(0, f(0)) = (0,0)$, $(2, f(2)) = (2,0)$, and $(1, f(1)) = (1,2)$, respectively. The area of this triangle is exactly 2.

(b) Here we shall *not* try to find the function x. (That would require knowledge of trigonometric and inverse trigonometric functions.) Instead we shall try to see if there is some other way to find the information we need in order to show that $t = 0$ is a minimum point for x. It turns out to be fairly easy in this case. In fact, $\dot{x}(t) < 0$ for $t < 0$ and $\dot{x}(t) > 0$ for $t > 0$. Hence, x is strictly decreasing in $(-\infty, 0]$ and strictly increasing in $[0, \infty)$, so $t = 0$ must be a global minimum point for $x = x(t)$. Note that this gives $x(t) \geq x(0) = 0$ for all t.

Furthermore,

$$\ddot{x} = \frac{d}{dt}\big((1+x^2)t\big) = 2x\dot{x}t + (1+x^2) = 2x(1+x^2)t^2 + (1+x^2).$$

Since $x(t) \geq 0$ for all t, we have $\ddot{x}(t) \geq 1 > 0$ for all t. It follows that x is (strictly) convex.

Exam problem 77

(i) We first calculate the indefinite integral. Integration by parts gives

$$\begin{aligned}\int x(2+x)^{1/3}\,dx &= x\frac{3}{4}(2+x)^{4/3} - \frac{3}{4}\int (2+x)^{4/3}\,dx \\ &= x\frac{3}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3} + C\end{aligned}$$

The definite integral is then

$$\begin{aligned}\int_{-1}^{6} x(2+x)^{1/3}\,dx &= \Big|_{-1}^{6} \left(\frac{3x}{4}(2+x)^{4/3} - \frac{9}{28}(2+x)^{7/3}\right) \\ &= \frac{9}{2}8^{4/3} - \frac{9}{28}8^{7/3} - \left(-\frac{3}{4} - \frac{9}{28}\right) = \frac{447}{14} \approx 31.92,\end{aligned}$$

where we have used that $8^{1/3} = \sqrt[3]{8} = 2$.

Alternatively, we can use substitution and calculate as follows: Introduce $u = (2+x)^{1/3}$ as a new variable. That gives $x = u^3 - 2$, $dx = 3u^2\,du$, and

$$\begin{aligned}\int x(2+x)^{1/3}\,dx &= \int (u^3-2)u3u^2\,du = \int (3u^6 - 6u^3)\,du \\ &= \frac{3}{7}u^7 - \frac{6}{4}u^4 + C = \frac{3}{7}(2+x)^{7/3} - \frac{3}{2}(2+x)^{4/3} + C.\end{aligned}$$

(This is indeed equal to the indefinite integral we found above, although it does not look that way at first glance.)

We the calculate the definite integral as before. However, we can also use formula (2) on page 333 in EMEA (page 355 in MA I). That will give us

$$\int_{-1}^{6} x(2+x)^{1/3} = \int_{1}^{2} (3u^6 - 6u^3)\,du = \Big|_{1}^{2} \left(\frac{3}{7}u^7 - \frac{3}{2}u^4\right)$$

etc.

(ii) Here we use the substitution $z = \sqrt[3]{x} = x^{1/3}$, which gives $x = z^3$ and $dx = 3z^2\,dz$. The integral then becomes

$$\int e^{\sqrt[3]{x}}\,dx = \int e^z 3z^2\,dz = 3\int z^2 e^z\,dz.$$

In order to find the last integral, we use integration by parts twice:

$$\begin{aligned}\int z^2e^z\,dz &= z^2e^z - \int 2ze^z\,dz = z^2e^z - \left(2ze^z - \int 2e^z\,dz\right) \\ &= z^2e^z - 2ze^z + \int 2e^z\,dz = z^2e^z - 2ze^z + 2e^z + C.\end{aligned}$$

Then

$$\int e^{\sqrt[3]{x}}\,dx = 3(z^2e^z - 2ze^z + 2e^z + C) = (3x^{2/3} - 6x^{1/3} + 6)e^{\sqrt[3]{x}} + C_1,$$

where $C_1 = 3C$.