Department of Economics
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ECON3120/4120 Mathematics 2, spring 2004
Problem solutions for seminar no. 6, 8–12 March 2004

(For practical reasons, some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 15.7.3 (= LA, 2.1.5)

Using the definitions of vector addition and multiplication of a vector by a real number, we get

$$3(x, y, z) + 5(-1, 2, 3) = (4, 1, 3) \iff (3x - 5, 3y + 10, 3z + 15) = (4, 1, 3)$$

Since two vectors are equal if and only if they are component-wise equal, this vector equation is equivalent to the equation system

$$3x - 5 = 4, \quad 3y + 10 = 1, \quad 3z + 15$$

with the obvious solution

$$x = 3, \quad y = -3, \quad z = -4.$$

EMEA, 15.7.4 (= LA, 2.1.6)

(a) For every vector $\mathbf{x}$ we have $\mathbf{x} + \mathbf{0} = \mathbf{x}$, so if $\mathbf{x} + \mathbf{0} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$, which means that all components of $\mathbf{x}$ are 0.

(b) Since $0\mathbf{x} = \mathbf{0}$ for every vector $\mathbf{x}$, this equation gives us *no information* about the components of $\mathbf{x}$.

EMEA, 15.7.6 (= LA, 2.1.9)

(Since we do not know the components of $\mathbf{a}$ and $\mathbf{b}$, this is an exercise in "abstract" vector manipulation.)

$$\begin{aligned} 4\mathbf{x} - 7\mathbf{a} = 2\mathbf{x} + 8\mathbf{b} - \mathbf{a} &\iff 4\mathbf{x} - 2\mathbf{x} = 8\mathbf{b} - \mathbf{a} + 7\mathbf{a} \\ &\iff 2\mathbf{x} = 6\mathbf{a} + 8\mathbf{b} \iff \mathbf{x} = 3\mathbf{a} + 4\mathbf{b}. \end{aligned}$$

EMEA, 15.7.8 (= LA, 2.2.4)

The dot product (inner product, scalar product, Norwegian: prikkproduktet, indreproduktet, skalarproduktet) is

$$(x, x - 1, 3) \cdot (x, x, 3x) = x^2 + (x - 1)x + 9x = 2x^2 + 8x = 2x(x + 4).$$

The two vectors are mutually orthogonal when their dot product equals 0, that is, for $x = 0$ and for $x = -4$.

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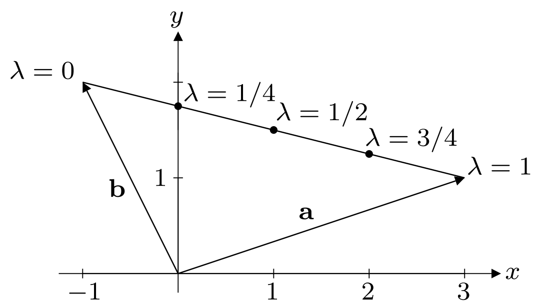
EMEA, 15.8.2 (= LA, 2.3.2)

(a) *Note:* In the EMEA version of this problem, $\mathbf{x} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$, whereas in LA, $\mathbf{x} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$. For convenience (= because of laziness), the illustration uses the definition from LA. EMEA readers please read λ as $1-\lambda$.



Problem 15.8.2

(b) As λ runs from 0 to 1, $\mathbf{x}$ will trace out the line segment from $\mathbf{a}$ to $\mathbf{b}$ in EMEA (from $\mathbf{b}$ to $\mathbf{a}$ in LA). By the point-point formula for a straight line ("topunktsformelen for en rett linje"), the straight line through $\mathbf{a} = (3, 1)$ and $\mathbf{b} = (-1, 2)$ has the equation

$$y - 1 = \frac{2-1}{-1-3}(x-3) = -\frac{1}{4}(x-3) \iff y = -\frac{1}{4}x + \frac{7}{4} \iff x + 4y = 7. \quad (*)$$

For every λ in $\mathbb{R}$ the point $(x, y) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} = (3-4\lambda, 1+\lambda)$ satisfies $(*)$, that is, the point lies on L. Conversely, if (x_0, y_0) is a point on L, then $x_0 = 7 - 4y_0$, and if we let $\lambda = y_0 - 1$, then $(1-\lambda)\mathbf{a} + \lambda\mathbf{b} = (3-4\lambda, 1+\lambda) = (7-4y_0, y_0) = (x_0, y_0)$.

EMEA 15.8.4 (= LA, 2.3.3)

(a) We get

$$x_1\mathbf{a} + x_2\mathbf{b} = (x_1, 2x_1, x_1) + (-3x_2, 0, -2x_2) = (x_1 - 3x_2, 2x_1, x_1 - 2x_2).$$

It follows that the vector equation $x_1\mathbf{a} + x_2\mathbf{b} = (5, 4, 4)$ is equivalent to the equation system

$$\begin{array}{ll} (1) & x_1 - 3x_2 = 5 \\ (2) & 2x_1 = 4 \\ (3) & x_1 - 2x_2 = 4 \end{array}$$

From (2), $\underline{x_1 = 2}$, and if we substitute this value for x_1 in (1), we get $2 - 3x_2 = 5$, which yields $\underline{x_2 = -1}$. Inserting these values for x_1 and x_2, we find that they also satisfy equation (3).

Note: This check is important, because it might well happen that there were no values of x_1 and x_2 that would satisfy all the equations (1)–(3). We see an example of this in part (b).

(b) The vector equation $x_1\mathbf{a} + x_2\mathbf{b} = (-3, 6, 1)$ yields the equation system

$$\begin{array}{lrcr} (1') & x_1 - 3x_2 &=& -3 \\ (2') & 2x_1 \qquad &=& 6 \\ (3') & x_1 - 2x_2 &=& 1 \end{array}$$

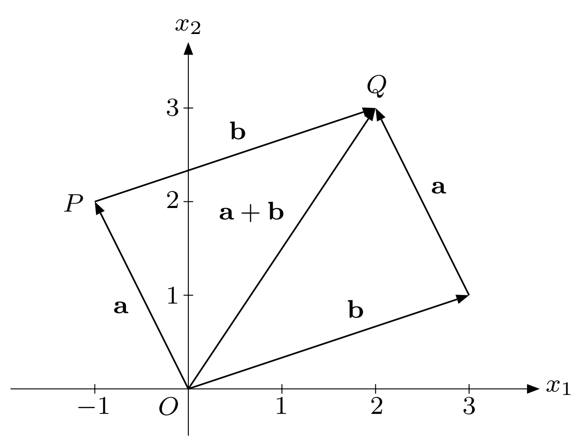
From $(2')$ we get $x_1 = 3$, and then $(1')$ yields $x_2 = 2$, but these values do not satisfy $(3')$. Hence there are no numbers x_1 and x_2 such that $x_1\mathbf{a}+x_2\mathbf{b} = (-3, 6, 1)$.

LA, 2.4.2

(a) The norms are

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{(-1)^2 + 2^2} = \sqrt{5}\,, \\ \|\mathbf{b}\| &= \sqrt{3^2 + 1^2} = \sqrt{10}\,, \\ \|\mathbf{a} + \mathbf{b}\| &= \|(2, 3)\| = \sqrt{2^2 + 3^2} = \sqrt{13}\,. \end{aligned}$$

Here $\|\mathbf{a}\|$, $\|\mathbf{b}\|$ og $\|\mathbf{a} + \mathbf{b}\|$ are the length of three sides OP, PQ og OQ of the triangle OQP in the figure. Since the length of each side in a proper triangle is less than the sum of the lengths of the other two, it is clear that $\|\mathbf{a}+\mathbf{b}\|$ must be less than $\|\mathbf{a}\| + \|\mathbf{b}\|$. (We would have equality if P were lying on the straight line segment from O to Q.)



Problem 2.4.2(a) in LA

(b) The dot product is $\mathbf{a} \cdot \mathbf{b} = (-1) \cdot 3 + 2 \cdot 1 = -1$, so $|\mathbf{a} \cdot \mathbf{b}| = 1$. The product of the norms is $\|\mathbf{a}\|\,\|\mathbf{b}\| = \sqrt{50} = 5\sqrt{2}$. Thus the Cauchy–Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\,\|\mathbf{b}\|$ does hold in this case.

LA, 2.4.5

(a) $(a, b) \cdot (-b, a) = -ab + ba = 0 \implies (a, b) \perp (-b, a)$.

(b) The dot product of the two vectors is 0 in this case too:

$$(a_1, a_2, a_3) \cdot (a_2 - a_3, a_3 - a_1, a_1 - a_2) = a_1(a_2 - a_3) + a_2(a_3 - a_1) + a_3(a_1 - a_2) = 0.$$

Exam problem 88

(a) (i) Integration by parts yields

$$\int 3xe^{-x/2}\,dx = 3x(-2e^{-x/2}) + \int 6e^{-x/2}\,dx = -6xe^{-x/2} - 12e^{-x/2} + C.$$

(ii) The substitution $u = 9 + \sqrt{x}$ gives $x = (u-9)^2$ and $dx = 2(u-9)\,du$. As x runs from 0 to 25, u runs from 9 to 14, and so we have

$$\int_0^{25} \frac{1}{9+\sqrt{x}}\,dx = \int_9^{14} \frac{2(u-9)}{u}\,du = \int_9^{14} \Big(2 - \frac{18}{u}\Big)\,du = 10 - 18\ln\frac{14}{9}\,.$$

(iii) We introduce $u = \sqrt{t+2}$ as a new variable and get $t = u^2 - 2$ and $dt = 2u\,du$. This yields

$$\begin{aligned}\int_2^7 t\sqrt{t+2}\,dt &= \int_2^3 (u^2-2)u\cdot 2u\,du = \int_2^3 (2u^4 - 4u^2)\,du \\ &= \Big|_2^3 \Big(\frac{2}{5}u^5 - \frac{4}{3}u^3\Big) = \Big(\frac{486}{5} - \frac{108}{3} - \frac{64}{5} + \frac{32}{3}\Big) = \frac{886}{15}\,.\end{aligned}$$

Exam problem 137

(a)
$$\begin{aligned}\int_4^9 \frac{(\sqrt{x}-1)^2}{x}\,dx &= \int_4^9 \frac{x - 2\sqrt{x}+1}{x}\,dx = \int_4^9 (1 - 2x^{-1/2} + \frac{1}{x})\,dx \\ &= \Big|_4^9 (x - 4\sqrt{x} + \ln x) = 9 - 4\cdot 3 + \ln 9 - 4 + 4\cdot 2 - \ln 4 = 1 + \ln\frac{9}{4}\end{aligned}$$

(b) We introduce the new variable $u = 1 + \sqrt{x}$. Then $u - 1 = \sqrt{x}$, $(u-1)^2 = x$, and $2(u-1)\,du = dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 2$. Hence

$$\begin{aligned}I &= \int_1^2 (\ln u)2(u-1)\,du = 2\int_1^2 u\ln u\,du - 2\int_1^2 \ln u\,du \\ &= 2\Big(\Big|_1^2 \frac{1}{2}u^2\ln u - \int_0^1 \frac{1}{2}u^2\frac{1}{u}\,du\Big) - 2\Big|_1^2 (u\ln u - u) \\ &= \Big|_1^2 (u^2\ln u - \frac{1}{2}u^2) - 2\Big|_1^2 (u\ln u - u) \\ &= 4\ln 2 - 2 - 0 + \frac{1}{2} - 2(2\ln 2 - 2 + 1) = \frac{1}{2}\end{aligned}$$

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