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Problem solutions for seminar no. 7, 15–19 March 2004

(For practical reasons, some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA 9.7.3 (= MA I, 10.9.3)

(a) Let us first use integration by parts to calculate the indefinite integral, using the fact that $\lambda e^{\lambda x} = (d/dx)(-e^{-\lambda x})$:

$$\int x\lambda e^{-\lambda x}\,dx = x(-e^{-\lambda x}) - \int 1\cdot e^{-\lambda x}\,dx = -xe^{-\lambda x} + \int e^{-\lambda x}\,dx$$
$$= -xe^{-\lambda x} - \frac{1}{\lambda}\,e^{-\lambda x} + C$$

This yields

$$\int_0^b x\lambda e^{-\lambda x}\,dx = \Big|_0^b -xe^{-\lambda x} - \frac{1}{\lambda}\,e^{-\lambda x}$$
$$= \Big(-be^{-\lambda b} - \frac{1}{\lambda}\,e^{-\lambda b}\Big) - \Big(-0 - \frac{1}{\lambda}e^{0}\Big) = \frac{1}{\lambda} - be^{-\lambda b} - \frac{1}{\lambda}\,e^{-\lambda b},$$

and so

$$\int_0^\infty x\lambda e^{-\lambda x}\,dx = \lim_{b\to\infty}\int_0^b x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}\,.$$

(We are assuming here that λ is a positive constant, as in Example 1 on page 335 (page 361 in MA I). We have also used that $be^{-\lambda b} \to 0$ as $b \to \infty$. This follows easily from using l'Hôpital's rule for "∞/∞"-expressions on $b/e^{\lambda b}$, or from the general result in formula (4) on page 264 (formula (4) on page 224 in MA I), with b instead of x.)

EMEA, 15.3.1 (= LA, 3.2.4)

See the answers in the book. Note that in (d), the product **AB** is not defined, since **A** is $2 \times \underline{2}$ and **B** is $\underline{3} \times 2$.

EMEA, 15.3.3 (= LA, 3.2.5)

See the answer in the back of the book. If you prefer to calculate $\mathbf{A}(\mathbf{BC})$ directly instead of using the associative law, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, then you will need the product

$$\mathbf{BC} = \begin{pmatrix} 14 & -4 & 10 \\ 21 & 0 & 27 \\ 11 & -4 & 13 \end{pmatrix}$$

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EMEA, 15.3.5 (= LA, 3.2.6)

(a) We know that $\mathbf{A}$ is an $m \times n$ matrix. Let $\mathbf{B}$ be a $p \times q$ matrix. The matrix product $\mathbf{AB}$ is defined if and only if $n = p$, and $\mathbf{BA}$ is defined if and only if $q = m$. So for both $\mathbf{AB}$ and $\mathbf{BA}$ to be defined, it is necessary and sufficient that $\mathbf{B}$ is an $n \times m$ matrix.

(b) (In LA, the matrix $\mathbf{B}$ is called $\mathbf{X}$.) We know from part (a) that if $\mathbf{BA}$ and $\mathbf{AB}$ are defined, then $\mathbf{B}$ must be a 2×2 matrix. So let $\mathbf{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then

$$\mathbf{BA} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} x + 2y & 2x + 3y \\ z + 2w & 2z + 3w \end{pmatrix}$$

and

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x + 2z & y + 2w \\ 2x + 3z & 2y + 3w \end{pmatrix}.$$

Hence,

$$\mathbf{BA} = \mathbf{AB} \quad \Longleftrightarrow \quad \begin{cases} x + 2y = x + 2z \\ 2x + 3y = y + 2w \\ z + 2w = 2x + 3z \\ 2z + 3w = 2y + 3w \end{cases}$$

The first and last of these four equations are true if and only if $y = z$, and if $y = z$, then the second and third are true if and only if $x = w - y$. Hence, the matrices $\mathbf{B}$ that commute with $\mathbf{A}$ are precisely the matrices of the form

$$\mathbf{B} = \begin{pmatrix} w - y & y \\ y & w \end{pmatrix} = w \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

where y and w can be any real numbers.

EMEA, 15.4.6 (= LA, 3.3.6)

(a) Matrix multiplication gives $\mathbf{AA} = \mathbf{A}$.

(b) By means of the equations

$$\text{(I)} \quad \mathbf{AB} = \mathbf{A} \qquad \text{and} \qquad \text{(II)} \quad \mathbf{BA} = \mathbf{B}$$

we get

$$\mathbf{AA} \overset{\text{(I)}}{=} (\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) \overset{\text{(II)}}{=} \mathbf{AB} = \mathbf{A}$$

and

$$\mathbf{BB} \overset{\text{(II)}}{=} (\mathbf{BA})\mathbf{B} = \mathbf{B}(\mathbf{AB}) \overset{\text{(I)}}{=} \mathbf{BA} = \mathbf{B}.$$

Hence, the matrices $\mathbf{A}$ and $\mathbf{B}$ are idempotent.

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EMEA, 15.5.4 (= LA, 3.5.4)

The matrix is symmetric if and only if the equations

$$a + 1 = a^2 - 1 \qquad \text{and} \qquad 4a = a^2 + 4$$

are *both* satisfied. Rearranging these equations gives

$$a^2 - a - 2 = 0 \qquad \text{and} \qquad a^2 - 4a + 4 = 0.$$

The roots of the first equation are $a = 2$ and $a = -1$, whereas the second has only the double root $a = 2$. Therefore the given matrix is symmetric if and only if $a = 2$.

EMEA, 15.6.1(b) (= LA, 4.1.1(a))

(b) We shall represent the equation system by its augmented (or extended) matrix, and perform elementary row operations on that:

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 5 \\ 2 & 3 & -1 & 1 \end{pmatrix} \begin{matrix} -1 \; -2 \\ \leftarrow\!\lrcorner \quad \\ \leftarrow\!\!-\!\!-\!\lrcorner \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & -3 & -7 \end{pmatrix} \begin{matrix} \\ -1/3 \\ \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & -1 & -3 & -7 \end{pmatrix} \begin{matrix} \\ 1 \\ \leftarrow\!\lrcorner \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & -3 & -22/3 \end{pmatrix} \begin{matrix} \\ \\ -1/3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix}$$

The last matrix here represents the equation system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 \qquad &= -1/3 \\ x_3 &= 22/9 \end{aligned}$$

and from this we can easily find the solution:

$$x_1 = \frac{20}{9}, \quad x_2 = -\frac{1}{3}, \quad x_3 = \frac{22}{9}.$$

Of course, as an alternative we could continue the Gaussian elimination process:

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix} \begin{matrix} \leftarrow\!\urcorner \\ -2 \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 1 & 14/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix} \begin{matrix} \leftarrow\!\urcorner \\ | \\ -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 20/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix}$$

Here the solution is immediately clear.

EMEA, 15.6.3 (= LA, 4.1.3)

We use Gaussian elimination (the variable names above the first matrix only serve to remind us which variables the coefficients belong to):

$$
\begin{array}{c}
\quad w \quad x \quad y \qquad z \\
\begin{pmatrix} 2 & 1 & 4 & 3 & 1 \\ 1 & 3 & 2 & -1 & 3c \\ 1 & 1 & 2 & 1 & c^2 \end{pmatrix}
\end{array}
\begin{matrix} -1/2 & 1/2 \\ \leftarrow & \\ \leftarrow & \end{matrix}
\sim \begin{pmatrix} 1 & 1/2 & 2 & 3/2 & 1/2 \\ 0 & 5/2 & 0 & -5/2 & 3c - 1/2 \\ 0 & 1/2 & 0 & -1/2 & c^2 - 1/2 \end{pmatrix}
\begin{matrix} \\ \leftarrow \\ \leftarrow \end{matrix}
$$

$$
\sim \begin{pmatrix} 1 & 1/2 & 2 & 3/2 & 1/2 \\ 0 & 1/2 & 0 & -1/2 & c^2 - 1/2 \\ 0 & 5/2 & 0 & -5/2 & 3c - 1/2 \end{pmatrix}
\begin{matrix} \leftarrow & & \\ -1 & -5 & 2 \\ & \leftarrow & \end{matrix}
$$

$$
\sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 - c^2 \\ 0 & 1 & 0 & -1 & 2c^2 - 1 \\ 0 & 0 & 0 & 0 & -5c^2 + 3c + 2 \end{pmatrix}
$$

The interchange of rows in step 2 is not necessary, but it helps us reduce the amount of calculation with fractions.

We can tell from the last matrix that the system has solutions if and only if $-5c^2 + 3c + 2 = 0$, that is, if and only if $\underline{c = 1 \text{ or } c = -2/5}$.

For these particular values of c we get

$$
\begin{aligned}
w &= 1 - c^2 - 2y - 2z = 1 - c^2 - 2a - 2b \\
x &= 2c^2 - 1 + z = 2c^2 - 1 + b \\
y &= a \\
z &= b
\end{aligned}
$$

where a and b are arbitrary numbers.

If we had not interchanged the rows in step 2 above, we would have ended with, among other things,

$$
w = \frac{3 - 3c}{5} - 2a - 2b \quad \text{and} \quad x = \frac{6c - 1}{5} - b,
$$

which looks quite different. But for the admissible values of c, that is, for $c = 1$ and for $c = -2/5$, this gives precisely the same answers as above, because we then have $c^2 = (3c + 2)/5$.

EMEA, 16.1.2 (= LA, 5.1.2)

See the answer in the back of the book. The area of the shaded parallelogram equals the area of the rectangle with corners in $(0, 0)$, $(3, 0)$, $(3, 6)$ and $(0, 6)$ (draw a picture!).

EMEA, 16.1.3 (= LA, 5.1.3)

(a) Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 8 & -1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{-16+5}{-6+1} = \frac{11}{5}, \qquad x_2 = \frac{\begin{vmatrix} 3 & 8 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{15-8}{-5} = -\frac{7}{5}.$$

EMEA, 16.1.6 (= LA, 5.1.6)

We have

$$\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C &= a, \end{aligned}$$

and if we use Cramer's rule on this equation system, we get

$$Y = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1-b},$$

$$C = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1-b}.$$

(It is also easy to solve this system without using Cramer's rule: From the first equation we get $Y = C + I_0 + G_0$ and if we substitute this expression for Y in $C = a + bY$, we get

$$\begin{aligned} C &= a + b(C + I_0 + G_0) \\ (1-b)C &= a + b(I_0 + G_0) \\ C &= \frac{a + b(I_0 + G_0)}{1-b} \end{aligned}$$

and so on.)

Exam problem 5

(a) Expansion along the first row gives

$$|\mathbf{A}| = \begin{vmatrix} a & b & 0 \\ -b & a & b \\ 0 & -b & a \end{vmatrix} = a \begin{vmatrix} a & b \\ -b & a \end{vmatrix} - b \begin{vmatrix} -b & b \\ 0 & a \end{vmatrix} = a(a^2+b^2) - b(-ab) = a^3 + 2ab^2.$$

Matrix multiplication gives

$$\mathbf{AA} = \begin{pmatrix} a^2-b^2 & 2ab & b^2 \\ -2ab & a^2-2b^2 & 2ab \\ b^2 & -2ab & a^2-b^2 \end{pmatrix}.$$

(b) We have

$$(\mathbf{C}'\mathbf{BC})' = \mathbf{C}'\mathbf{B}'(\mathbf{C}')' = \mathbf{C}'(-\mathbf{B})\mathbf{C} = -\mathbf{C}'\mathbf{BC}.$$

(c) Since $\mathbf{A}' = \begin{pmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{pmatrix}$, the matrix $\mathbf{A}$ is skew-symmetric, that is, $\mathbf{A}' = -\mathbf{A}$, if and only if $a = 0$.

Exam problem 52

(a) Direct calculation yields

$$\mathbf{AA}' = \begin{pmatrix} 21 & 11 \\ 11 & 10 \end{pmatrix}, \qquad |\mathbf{AA}'| = 89, \qquad (\mathbf{AA}')^{-1} = \frac{1}{89}\begin{pmatrix} 10 & -11 \\ -11 & 21 \end{pmatrix}.$$

(b) No, it is no coincidence. For any matrix $\mathbf{A}$, the product $\mathbf{AA}'$ is symmetric, since

$$(\mathbf{AA}')' = (\mathbf{A}')'\mathbf{A}' = \mathbf{AA}'.$$

Furthermore, the inverse of a symmetric matrix is always symmetric.

(c) Since $\mathbf{1}$ is a $1 \times m$ matrix and $\mathbf{X}$ is $m \times n$, the product $\frac{1}{m}\mathbf{1} \cdot \mathbf{X}$ is a $1 \times n$ matrix, that is, a row vector of dimension n. We get

$$\frac{1}{m}\mathbf{1} \cdot \mathbf{X} = \frac{1}{m}(1, 1, \dots, 1) \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} =$$

$$\frac{1}{m}(x_{11} + x_{21} + \dots + x_{m1},\; x_{12} + x_{22} + \dots + x_{m2},\; \dots,\; x_{1n} + x_{2n} + \dots + x_{mn})$$

and the ith component of this vector, $\dfrac{x_{1i} + x_{2i} + \dots + x_{mi}}{m}$, is *the arithmetic mean* of the m observations of quantity no. i.