Department of Economics
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Problem solutions for seminar no. 8, 22–26 March 2004

(For practical reasons, some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 16.3.2 (= LA, 5.3.2(a))

We get $+a_{12}a_{23}a_{35}a_{41}a_{54}$. (Four of the lines that connect pairs of elements are rising to the right.)

EMEA, 16.4.2 (= LA, 5.4.2)

Direct calculation gives

$$\begin{aligned}
|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 5 \end{vmatrix} &= 2\cdot 0\cdot 5 - 2\cdot 1\cdot 2 + 1\cdot 1\cdot 1 - 1\cdot 1\cdot 5 + 3\cdot 1\cdot 2 - 3\cdot 0\cdot 1 \\
&= 0 - 4 + 1 - 5 + 6 - 0 = -2, \\
|\mathbf{A}'| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 5 \end{vmatrix} &= 2\cdot 0\cdot 5 - 2\cdot 2\cdot 1 + 1\cdot 2\cdot 3 - 1\cdot 1\cdot 5 + 1\cdot 1\cdot 1 - 1\cdot 0\cdot 3 \\
&= 0 - 4 + 6 - 5 + 1 - 0 = -2.
\end{aligned}$$

EMEA, 16.4.6 (= LA, 5.4.4)

This problem is an exercise in using some of the rules in Theorem 16.4.1 in EMEA (LA: setning 5.1).

(a) This determinant is zero because the second column equals 2 times the first column. We could also have used that the sum if the first and the second row equals the third row.

(b) By adding the second column to the third, we get

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = 0$$

The last inequality follows because the third column is proportional to the first.

v04ste08 1.3.2004 1051

(c) There is a common factor $x - y$ in the first row, so we get

$$\begin{vmatrix} x-y & x-y & x^2-y^2 \\ 1 & 1 & x+y \\ y & 1 & x \end{vmatrix} = (x-y)\begin{vmatrix} 1 & 1 & x+y \\ 1 & 1 & x+y \\ y & 1 & x \end{vmatrix} = 0,$$

since the new determinant has two equal rows.

EMEA, 16.5.2 (= LA, 5.5.2)

(a) Cofactor expansion along the first row gives

$$\begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix} = 0 - 0 + a\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = a \cdot (-bc) = -abc.$$

(b) Cofactor expansion again:

$$\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{vmatrix} = 0 - 0 + 0 - a\begin{vmatrix} 0 & 0 & b \\ 0 & c & 0 \\ d & 0 & 0 \end{vmatrix} = -a \cdot (-bcd) = abcd,$$

by the result from part (a).

(c) Repeated cofactor expansion:

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 4 & 0 & 3 & 4 \\ 6 & 2 & 3 & 1 & 2 \end{vmatrix} = 0 - 0 + 0 - 0 + 1\begin{vmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 4 & 0 & 3 \\ 6 & 2 & 3 & 1 \end{vmatrix}$$

$$= 0 - 0 + 0 - 5\begin{vmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 6 & 2 & 3 \end{vmatrix} = -5 \cdot 3\begin{vmatrix} 0 & 4 \\ 6 & 2 \end{vmatrix} = -5 \cdot 3 \cdot (-4 \cdot 6) = 360.$$

EMEA, 16.6.2 (= LA, 6.1.2)

It suffices to show (by direct calculation) that the product of the two matrices is $\mathbf{I}_3$.

v04ste08 1.3.2004 1051

EMEA, 16.7.1 (= LA, 6.2.1)

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$. The adjoint matrix is

$$\operatorname{adj} \mathbf{A} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix},$$

and the determinant is

$$|\mathbf{A}| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = 1 \cdot 1 + 2 \cdot 4 + 0 \cdot 2 = 9$$

(by expansion along the first column). Hence,

$$\mathbf{A}^{-1} = \frac{1}{9}(\operatorname{adj} \mathbf{A}) = \frac{1}{9} \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}.$$

EMEA, 16.8.2 (= LA, 6.3.2)

The coefficient matrix of the system and its determinant are

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}, \qquad |\mathbf{A}| = -10.$$

Since $\mathbf{A} \neq 0$, the system has a unique solution for any set of right-hand sides. The determinants D_1, D_2, and D_3 of the matrices that we get by replacing the first, second, and third column in $\mathbf{A}$ with the right-hand sides are

$$D_1 = \begin{vmatrix} b_1 & 1 & 0 \\ b_2 & -1 & 2 \\ b_3 & 3 & -1 \end{vmatrix} = -5b_1 + b_2 + 2b_3\,,$$

$$D_2 = \begin{vmatrix} 3 & b_1 & 0 \\ 1 & b_2 & 2 \\ 2 & b_3 & -1 \end{vmatrix} = 5b_1 - 3b_2 - 6b_3\,,$$

$$D_3 = \begin{vmatrix} 3 & 1 & b_1 \\ 1 & -1 & b_2 \\ 2 & 3 & b_3 \end{vmatrix} = 5b_1 - 7b_2 - 4b_3\,.$$

It is probably most convenient to calculate each of these determinants by cofactor expansion along the column containing the bs.

By Cramer's rule, the solution of the system is

$$\begin{aligned}
x_1 &= \frac{D_1}{|\mathbf{A}|} = \frac{5b_1 - b_2 - 2b_3}{10} = \frac{1}{2}b_1 - \frac{1}{10}b_2 - \frac{1}{5}b_3\,, \\
x_2 &= \frac{D_2}{|\mathbf{A}|} = \frac{-5b_1 + 3b_2 + 6b_3}{10} = -\frac{1}{2}b_1 + \frac{3}{10}b_2 + \frac{3}{5}b_3\,, \\
x_3 &= \frac{D_3}{|\mathbf{A}|} = \frac{-5b_1 + 7b_2 + 4b_3}{10} = -\frac{1}{2}b_1 + \frac{7}{10}b_2 + \frac{2}{5}b_3\,.
\end{aligned}$$

Exam problem 37

(a) From the definition of a 3×3 determinant we get

$$|\mathbf{A}| = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0 \cdot 1 \cdot 1 - 0 \cdot 1 \cdot 0 + 1 \cdot 1 \cdot 1 - 1 \cdot 0 \cdot 1 + 0 \cdot 0 \cdot 0 - 0 \cdot 1 \cdot 1 = 1.$$

(A more efficient procedure would be to use cofactor expansion along the first row or the first column.) Matrix multiplication yields

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \\
\mathbf{A}^3 &= \mathbf{A} \cdot \mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}
\end{aligned}$$

Then, by direct calculation we get $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$.

(b) Simple matrix calculation yields

$$\begin{aligned}
\mathbf{A}(\mathbf{A} - \mathbf{I})^2 &= \mathbf{A}(\mathbf{A} - \mathbf{I})(\mathbf{A} - \mathbf{I}) = \mathbf{A}(\mathbf{A}^2 - \mathbf{I} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{I} + \mathbf{I}^2) \\
&= \mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}) = \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A},
\end{aligned}$$

and by the last result in part (a), $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} = \mathbf{I}$. Hence, $\mathbf{A}(\mathbf{A} - \mathbf{I})^2 = \mathbf{I}$, and $(\mathbf{A} - \mathbf{I})^2$ is therefore the inverse matrix of $\mathbf{A}$.

Exam problem 45

Direct calculation shows that

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ a + b & 2a + 1/4 + 3b & 4a + 3/2 + 2b \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $\mathbf{A}$ is the inverse of $\mathbf{B}$ if and only if $\mathbf{AB} = \mathbf{I}_3$, that is, if and only if

$$a + b = 0, \quad 2a + 3b = 3/4, \quad 4a + 2b = -3/2.$$

The first of these equations gives $b = -a$, and if we substitute $-a$ for b in the second equation, we get $-a = 3/4$, so $a = -3/4$ and $b = 3/4$. These values of a and b also satisfy the last equation. (It is important to check that!)

Exam problem 50

(a) Expand the determinant of the system along the first row:

$$\begin{aligned}\begin{vmatrix} 1 & 1 & -1 \\ k & 3 & -2 \\ 6 & 2k & -3k \end{vmatrix} &= \begin{vmatrix} 3 & -2 \\ 2k & -3k \end{vmatrix} - \begin{vmatrix} k & -2 \\ 6 & -3k \end{vmatrix} + (-1)\begin{vmatrix} k & 3 \\ 6 & 2k \end{vmatrix} \\ &= (-9k + 4k) - (-3k^2 + 12) - (2k^2 - 18) \\ &= k^2 - 5k + 6 = (k-2)(k-3).\end{aligned}$$

The system has a unique solution if and only if the determinant is $\neq 0$, that is, if and only if $k \neq 2$ and $k \neq 3$.

(b) For $k = 3$ we have the system

$$\begin{array}{rl} x + y - z = 2 & \quad -3 \quad -6 \\ 3x + 3y - 2z = 1 & \\ 6x + 6y - 9z = 0 & \end{array}$$

With the elementary row operations indicated, we get the equation system

$$\begin{aligned} x + y - z &= 2 \\ z &= -5 \\ -3z &= -12 \end{aligned}$$

which has no solution (the last two equations are contradict each other).

We could also formulate the argument in an apparently simpler fashion, without explicitly mentioning elementary operations: From the first equation in the original system we get $x + y = z + 2$. The other two equations then give

$$\begin{aligned} 3(z+2) - 2z &= 1 \\ 6(z+2) - 9z &= 0 \end{aligned} \qquad \text{which in turn gives} \qquad \begin{aligned} z &= -5 \\ -3z &= -12 \end{aligned}$$

and this system obviously has no solution. (But of course this is really the same argument as above.)

Exam problem 62

(a) The determinant of $\mathbf{A}_a$ is

$$\begin{aligned} |\mathbf{A}_a| = \begin{vmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & -a \\ -a & 1 \end{vmatrix} - (-a)\begin{vmatrix} -a & -a \\ -a & 1 \end{vmatrix} + (-a)\begin{vmatrix} -a & 1 \\ -a & -a \end{vmatrix} \\ &= -2a^3 - 3a^2 + 1. \end{aligned}$$

It is easy to see that $-2a^3 - 3a^2 + 1 = 0$ for $a = 1/2$. Hence, $a - 1/2$ is a factor in $-2a^3 - 3a^2 + 1$. Polynomial division gives

$$(-2a^3 - 3a^2 + 1) \div (a - 1/2) = -2a^2 - 4a - 2 = -2(a+1)^2,$$

so $|\mathbf{A}_a| = -2(a+1)^2(a - 1/2)$, and

$$|\mathbf{A}_a| \neq 0 \iff a \neq -1 \text{ and } a \neq 1/2.$$

Thus, $\mathbf{A}_a$ has an inverse precisely when a is different from -1 and $1/2$.

Note: In English, division of a by b is usually written as $a \div b$ rather than $a : b$.

(b) Let $\mathbf{B} = k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix}$. The product of $\mathbf{A}_a$ and $\mathbf{B}$ is

$$\begin{aligned}
\mathbf{A}_a\mathbf{B} &= \begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix} \cdot k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix} \\
&= k \begin{pmatrix} 1-a-2a^2 & 0 & 0 \\ 0 & 1-a-2a^2 & 0 \\ 0 & 0 & 1-a-2a^2 \end{pmatrix} \\
&= k(1-a-2a^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= k(1-a-2a^2)\,\mathbf{I}_3 = k(1+a)(1-2a)\,\mathbf{I}_3.
\end{aligned}$$

This shows that if we choose $k = 1/(1-a-2a^2)$, then $\mathbf{B}$ is the inverse of $\mathbf{A}_a$, and this works for all values of a except -1 and $1/2$.

(c) Let $\mathbf{x} = (x_1, x_2, x_3)'$. Then

$$\mathbf{A}_a^{-1}\mathbf{x} = k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} (1-a)x_1 + ax_2 + ax_3 \\ ax_1 + (1-a)x_2 + ax_3 \\ ax_1 + ax_2 + (1-a)x_3 \end{pmatrix}.$$

If $0 < a < 1/2$, then k is positive, and then all components of $\mathbf{A}_a^{-1}\mathbf{x}$ are positive if the components x_1, x_2, and x_3 of $\mathbf{x}$ are positive.

(Actually, it is sufficient to note that the elements of $\mathbf{A}_a^{-1}$ are positive when $a \in (0, 1/2)$. Then it follows directly from the definition of matrix multiplication that all components of $\mathbf{A}_a^{-1}\mathbf{x}$ will be positive when the components of $\mathbf{x}$ are positive. Note that if $a \in (1/2, 1)$, then all elements of $\mathbf{A}_a$ will be *negative*, since $k < 0$ for these values of a.)

v04ste08 1.3.2004 1051