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April 2004
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ECON3120/4120 Mathematics 2, spring 2004
Problem solutions for seminar no. 10, 19–23 April 2004

(For practical reasons, some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 12.5.3 (= MA I, 11.12.3)

This ought to be straightforward. See the answers in the back of the book.

EMEA, 12.6.1 (= MA I, 11.13.1)

(c) Homogeneous of degree $-1/2$, because

$$h(tx, ty, tz) = \frac{\sqrt{tx} + \sqrt{ty} + \sqrt{tz}}{tx + ty + tz} = \frac{\sqrt{t}\,(\sqrt{x} + \sqrt{y} + \sqrt{z}\,)}{t(x + y + z)} = \frac{1}{\sqrt{t}} h(x, y, z)$$

for all $t > 0$.

(e) Not homogeneous. For $t > 0$ we get

$$\frac{H(tx, ty)}{H(x, y)} = \frac{\ln x + \ln y + 2 \ln t}{\ln x + \ln y} = 1 + \frac{2 \ln t}{H(x, y)}\,,$$

which is not a power of t.

EMEA, 7.1.3 (= MA I, 7.1.3)

(a) We differentiate with respect to x on both sides of the equation

$$2x^2 + 6xy + y^2 = 18,$$

remembering that y is a function of x. That gives

$$4x + 6y + 6xy' + 2yy' = 0,$$

$$(6x + 2y)y' = -4x - 6y,$$

$$(*) \qquad y' = -\frac{4x + 6y}{6x + 2y} = -\frac{2x + 3y}{3x + y}\,.$$

At the point $(1, 2)$ we get

$$y' = -\frac{2 \cdot 1 + 3 \cdot 2}{3 \cdot 1 + 2} = -\frac{8}{5}\,.$$

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(b) From (∗) in part (a) we get

$$y'' = -\frac{(2+3y')(3x+y) - (2x+3y)(3+y')}{(3x+y)^2}\,.$$

Then, if we let $x = 1$, $y = 2$, and $y' = -8/5$, we get $y'' = \cdots = \dfrac{126}{125}$.

EMEA, 7.1.7 (= MA I, 7.1.8)

(a) Using the expressions for C and M from (ii) and (iii) in (i), we get

$$Y = f(Y) + I + \bar{X} - g(Y),$$

hence,

$$Y - f(Y) + g(Y) = I + \bar{X}.$$

If we assume that this equation defines Y as a function of I, then implicit differentiation gives

$$(1 - f'(Y) + g'(Y))\frac{dY}{dI} = 1,$$

that is,

$$\frac{dY}{dI} = \frac{1}{1 - f'(Y) + g'(Y)}\,.$$

If we now assume that $0 < f'(Y) < 1$ and $g'(Y) > 0$, we get $\dfrac{dY}{dI} > 0$.

(b)
$$\begin{aligned}
\frac{d^2Y}{dI^2} &= \frac{d}{dI}\left(\frac{dY}{dI}\right) = \frac{d}{dY}\left(\frac{dY}{dI}\right)\cdot\frac{dY}{dI} \\
&= \frac{d}{dY}\left(\frac{1}{1 - f'(Y) + g'(Y)}\right)\cdot\frac{1}{1 - f'(Y) + g'(Y)} \\
&= -\frac{-f''(Y) + g''(Y)}{(1 - f'(Y) + g'(Y))^2}\cdot\frac{1}{1 - f'(Y) + g'(Y)} \\
&= \frac{f''(Y) - g''(Y)}{(1 - f'(Y) + g'(Y))^3}\,.
\end{aligned}$$

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Exam problem 29

(a) $$\frac{\partial E}{\partial p} = -aAp^{-a-1}r^b, \qquad \frac{\partial E}{\partial r} = bAp^{-a}r^{b-1}$$

gives

$$p\frac{\partial E}{\partial p} + r\frac{\partial E}{\partial r} = -aAp^{-a}r^b + bAp^{-a}r^b = (b-a)Ap^{-a}r^b = kE$$

if we let $k = b-a$. *Note that this is also a consequence of Euler's theorem, since E is homogeneous of degree $b-a$.* With $a = 1.5$ and $b = 2$, we get $k = b - a = 0.58$.

(b) The second-order partial derivatives are

$$\begin{aligned}
\frac{\partial^2 E}{\partial p^2} &= (-a-1)(-a)Ap^{-a-2}r^b = (a+1)a\frac{E}{p^2}\,, \\
\frac{\partial^2 E}{\partial p \partial r} &= -abAp^{-a-1}r^{b-1} = -ab\frac{E}{pr}\,, \\
\frac{\partial^2 E}{\partial r^2} &= (b-1)bAp^{-a}r^{b-2} = (b-1)b\frac{E}{r^2}\,.
\end{aligned}$$

A little calculation then shows that

$$\begin{aligned}
p^2\frac{\partial^2 E}{\partial p^2} + 2pr\frac{\partial^2 E}{\partial p \partial r} + r^2\frac{\partial^2 E}{\partial r^2} &= (a+1)aE - 2abE + (b-1)bE \\
&= (a^2 + a - 2ab + b^2 - b)E \\
&= ((a-b)^2 + a - b)E \\
&= (a-b)(a-b+1)E.
\end{aligned}$$

Of course, this is also a special case of formula (5) on page 433 in EMEA (page 412 in MA I).

(c) We get

$$\frac{dE}{dt} = \frac{\partial E}{\partial p}\frac{dp}{dt} + \frac{\partial E}{\partial r}\frac{dr}{dt} = -\frac{a}{p}E\frac{dp}{dt} + \frac{b}{r}E\frac{dr}{dt} = \left(-\frac{a}{p}\frac{dp}{dt} + \frac{b}{r}\frac{dr}{dt}\right)E.$$

With $p(t) = p_0(1.06)^t$ and $r(t) = r_0(1.08)^t$, we have

$$\begin{aligned}
\frac{dp}{dt} &= p_0(1.06)^t \ln 1.06, &\qquad \frac{a}{p}\frac{dp}{dt} &= a\ln 1.06, \\
\frac{dr}{dt} &= r_0(1.06)^t \ln 1.08, &\qquad \frac{b}{r}\frac{dr}{dt} &= b\ln 1.08,
\end{aligned}$$

which gives

$$\frac{dE}{dt} = (b\ln 1.08 - a\ln 1.06)E = \ln\frac{(1.08)^b}{(1.06)^a}\cdot Ap_0^{-a}(1.06)^{-at}r_0^b(1.08)^{bt}$$
$$= \ln Q\cdot E(p_0,r_0)\left(\frac{(1.08)^b}{(1.06)^a}\right)^t = E(p_0,r_0)Q^t\ln Q.$$

(d) E increases with t if and only if $dE/dt \geq 0$, and from the above we see that

$$\frac{dE}{dt}\geq 0 \iff b\ln 1.08 - a\ln 1.06 \geq 0 \iff \frac{b}{a}\geq\frac{\ln 1.06}{\ln 1.08}\ (\approx 0.757122).$$

Exam problem 31

(a) (i)
$$f(tx_1,tx_2) = 5(tx_1)^4 + 6(tx_1)(tx_2)^3 = 5t^4x_1^4 + 6tx_1t^3x_2^3$$
$$= t^4(tx_1^4 + 6x_1x_2^3 = t^4f(x_1,x_2),$$

so f is homogeneous of degree 4.

(ii) If F were homogeneous of degree k, then

$$F(t,t,t) = F(t\cdot 1, t\cdot 1, t\cdot 1) = t^kF(1,1,1),$$

but $F(t,t,t) = e^{3t}$ and $t^kF(1,1,1) = t^ke^3$, which is not the same as e^{3t} for any k. (The equality would have to hold for some *constant* k, and for all $t > 0$.)

(iii) $G(tK,tL,tM,tN) = \cdots = G(K,L,M,N) = t^0G(K,L,M,N)$, so G is homogeneous of degree 0.

(b)
$$x_1f_1'(x_1,x_2) + x_2f_2'(x_1,x_2) = x_1(20x_1^3 + 6x_2^3) + x_2 18x_1x_2^2$$
$$= 20x_1^4 + 24x_1x_2^3 = 4f(x_1,x_2),$$

in accordance with Euler's theorem.

Exam problem 99

Implicit differentiation with respect to x yields

$$y + xy' - \frac{e^{xy}(y+xy') - e^{-xy}(y+xy')}{e^{xy}+e^{-xy}} = 0.$$

Rearranging this equation yields

$$2ye^{-xy} + 2xe^{-xy}y' = 0 \implies y' = -\frac{y}{x}\,.$$

It then follows that

$$y'' = \frac{d}{dx}y' = -\frac{d}{dx}\frac{y}{x} = -\frac{y'x-y}{x^2} = -\frac{-y-y}{x^2} = \frac{2y}{x^2}\,.$$

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It may be worthwhile to note that

$$F(x,y) = 1 + xy - \ln(e^{xy} + e^{-xy}) = g(xy),$$

where

$$g(u) = 1 + u - \ln(e^u + e^{-u}).$$

The function g is strictly increasing because

$$g'(u) = 1 - \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^{-u}}{e^u + e^{-u}} > 0$$

for all u. Therefore g has an inverse function, and

$$F(x,y) = c \iff g(xy) = c \iff xy = g^{-1}(c).$$

This means that a level curve for F is also a level curve for xy, and we could have used this to find y' and y'' with less work.

NOTE! In any case it is a bad idea to use the formula

$$y'' = -\frac{1}{(F_2')^3}\left[F_{11}''(F_2')^2 - 2F_{12}''F_1'F_2' + F_{22}''(F_1')^2\right] = \frac{1}{(F_2')^3}\begin{vmatrix} 0 & F_1' & F_2' \\ F_1' & F_{11}'' & F_{12}'' \\ F_2' & F_{21}'' & F_{22}'' \end{vmatrix}$$

on page 425 in EMEA (page 426 in MA I). If we have managed to find an expression for y' in terms of x and y, it is almost always better to use that expression in order to find y''.

Extra problems:

Exam problem 28

(a) The first and second derivatives of g are

$$\begin{aligned} g'(x) &= 2 + ae^{-x}(1+x^2) - 2axe^{-x} \\ &= 2 + ae^{-x}(1 + x^2 - 2x) = 2 + ae^{-x}(x-1)^2, \\ g''(x) &= -ae^{-x}(1+x^2) + 2axe^{-x} - 2ae^{-x} + 2axe^{-x} \\ &= -ae^{-x}(1 + x^2 - 2x + 2 - 2x) = -ae^{-x}(x^2 - 4x + 3) \\ &= -ae^{-x}(x-1)(x-3). \end{aligned}$$

With a sign diagram or otherwise we find that

$$g''(x) \begin{cases} < 0 & \text{if } x < 1, \\ = 0 & \text{if } x = 1, \\ > 0 & \text{if } 1 < x < 3, \\ = 0 & \text{if } x = 3, \\ < 0 & \text{if } x > 3. \end{cases}$$

It follows that g is concave in $(-\infty, 1]$, convex in $[1,3]$ and concave in $[3,\infty)$.

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(b) Since $\lim_{x\to\infty} e^{-x}(1+x^2) = \lim_{x\to\infty} \dfrac{1+x^2}{e^x} = 0$, we see that

$$g(x) = 2x - ae^{-x}(1+x^2) \to \infty \quad \text{as} \quad x \to \infty.$$

This means, in particular, that $g(x) > 0$ for sufficiently large values of x. Furthermore, $g(0) = -a < 0$, so the intermediate value theorem ("skjæringssetningen") guarantees that g has at least one zero in $(0, \infty)$. It follows from part (a) that $g'(x) = 2 + ae^{-x}(1-x)^2 > 0$, so g is strictly increasing throughout $(-\infty, \infty)$. Hence, g cannot have more than one zero. It follows that g has exactly one zero, x_0, and that $x_0 > 0$.

(c) Since $g'(x) > 2$ for $x \neq 1$, the graph of g over $(0, \infty)$ must lie above the straight line that passes through $(0, -a)$ and has slope 2. (Recall that $g(0) = -a$.) This straight line intersects the x-axis for $x = a/2$, and it follows that $g(a/2) > 0$. (Draw a picture!) Hence, $0 < x_0 < a/2$.

(d) The derivative of $f(x) = ae^{-x} + \ln(1+x^2)$ is

$$f'(x) = -ae^{-x} + \frac{2x}{1+x^2} = \frac{g(x)}{1+x^2}\,.$$

This shows that $f'(x)$ has the same sign as $g(x)$, and therefore

$$f'(x) < 0 \text{ for } x < x_0, \quad f'(x) = 0 \text{ for } x = x_0, \quad f'(x) > 0 \text{ for } x > x_0.$$

Hence, the function f is decreasing in $(-\infty, x_0]$ and increasing in $[x_0, \infty)$, and it follows that x_0 is a global minimum point of f.

(e) We know that $g(x_0) = 2x_0 - ae^{-x_0}(1+x_0^2) = 0$. This equation defines x_0 as a function of a, and implicit differentiation gives

$$\frac{dx_0}{da} = -\frac{-e^{-x_0}(1+x_0^2)}{g'(x_0)} = \frac{1+x_0^2}{e^{x_0}g'(x_0)} = \frac{1+x_0^2}{2e^{x_0} + a(x_0-1)^2}\,.$$

(f) We have $0 < x_0 < a/2$, and therefore x_0 will tend to 0 as $a \to 0^+$. It follows from l'Hôpital's rule that

$$\lim_{a\to 0^+} \frac{x_0}{a} = \text{“}\frac{0}{0}\text{”} = \lim_{a\to 0^+} \frac{dx_0/da}{1} = \lim_{a\to 0^+} \frac{1+x_0^2}{2e^{x_0} + a(x_0-1)^2} = \frac{1+0^2}{2e^0+0} = \frac{1}{2}\,.$$

Exam problem 59

(a) Direct calculation yields

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix}$$

$$\mathbf{I}_3 - \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -10 & -5 & 1 \end{pmatrix}$$

and, finally,

$$(\mathbf{I}_3 - \mathbf{A})(\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2) = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -10 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3.$$

(b) By the last result in (a),

$$(\mathbf{I}_3 - \mathbf{A})^{-1} = \mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix}.$$

(c) $(\mathbf{I}_n + a\mathbf{U})(\mathbf{I}_n + b\mathbf{U}) = \mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + ab\mathbf{U}^2$. But

$$\mathbf{U}^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{pmatrix} = n\mathbf{U},$$

and consequently,

$$\mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + ab\mathbf{U}^2 = \mathbf{I}_n + a\mathbf{U} + b\mathbf{U} + nab\mathbf{U} = \mathbf{I}_n + (a + b + nab)\mathbf{U}.$$

(d) Let us call the given matrix $\mathbf{D}$. It is easy to see that

$$\mathbf{D} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \mathbf{I}_3 + 3\mathbf{U}.$$

Let b be an arbitrary number. From the result in (c), we get

$$(\mathbf{I}_3 + 3\mathbf{U})(\mathbf{I}_3 + b\mathbf{U}) = \mathbf{I}_3 + (3 + b + 3 \cdot 3b\mathbf{U}) = \mathbf{I}_3 + (3 + 10b)\mathbf{U}.$$

If we choose $b = -3/10$, then the last matrix expression above equals $\mathbf{I}_3$, and it follows that

$$\mathbf{D}^{-1} = (\mathbf{I}_3 + 3\mathbf{U})^{-1} = \mathbf{I}_3 - (3/10)\mathbf{U}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3/10 & 3/10 & 3/10 \\ 3/10 & 3/10 & 3/10 \\ 3/10 & 3/10 & 3/10 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}.$$

Exam problem 124

(a) By elementary operations on the augmented matrix we find that it is equivalent to:

$$\begin{pmatrix} 1 & 1 & 1 & q \\ 0 & 1 & 2 & -p+q \\ 0 & 0 & p-3 & 5+p-q-p^2 \end{pmatrix}$$

The last row represents the equation

$$(p-3)x_3 = 5 + p - q - p^2$$

If $p \neq 3$, then x_3 is uniquely determined, and then x_1 and x_2 are also uniquely determined. If $p = 3$ and $5 + p - q - p^2 = 5 + 3 - q - 9 = -q - 1 \neq 0$, that is, if $p = 3$ and $q \neq -1$, there is no solution. If $p = 3$ and $q = -1$, the last equation becomes $0 = 0$. In this case we can choose x_3 freely, and the system has 1 degree of freedom.

(b) With $p = 3$, $q = -1$ we get $x_1 + x_2 + x_3 = -1$ and $x_2 + 2x_3 = -4$. We can choose $x_3 = t$ freely, and get $x_1 = t + 3$, $x_2 = -2t - 4$.

(c) $$\begin{vmatrix} 31 & 32 & 33 \\ 32 & 33 & 35 \\ 33 & 34 & 36 \end{vmatrix} \begin{matrix} -1 & -1 \\ \leftarrow\!\lrcorner & \\ \leftarrow & \lrcorner \end{matrix} = \begin{vmatrix} 31 & 32 & 33 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{vmatrix} \begin{matrix} \\ -2 \\ \leftarrow\!\lrcorner \end{matrix} = \begin{vmatrix} 31 & 32 & 33 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$
$$= (-1)(31 - 32) = 1.$$

(d) $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3) = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3-\mathbf{A}-\mathbf{A}^2-\mathbf{A}^3-\mathbf{A}^4 = \mathbf{I}-\mathbf{A}^4 = \mathbf{I}$. Hence, $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = (\mathbf{I} - \mathbf{A})^{-1}$. (See (4) on page 610 in EMEA, (6.4) on page 111 in LA.)