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Problem solutions for seminar no. 11, 26–30 April 2004

(For practical reasons, some of the solutions may include problem parts that were not on the problem list for the seminar.)

EMEA, 7.3.8 (= MA I, 7.3.8)

(b) From $f(x) = 1/x$ and $f'(x) = -1/x^2$ we get

$$\begin{aligned}\Delta y &= f(x+dx) - f(x) = \frac{1}{x+dx} - \frac{1}{x} = -\frac{dx}{x(x+dx)}\,, \\ dy &= f'(x)\,dx = -\frac{dx}{x^2}\,.\end{aligned}$$

(i) With $x = 3$ and $dx = -1/10$,

$$\begin{aligned}\Delta y &= -\frac{-\frac{1}{10}}{3\big(3-\frac{1}{10}\big)} = \frac{1}{3\cdot 29} = \frac{1}{87} \approx 0.01149, \\ dy &= -\frac{-\frac{1}{10}}{3^2} = \frac{1}{90} \approx 0.01111.\end{aligned}$$

(ii) With $x = 3$ and $dx = -1/100$,

$$\begin{aligned}\Delta y &= -\frac{-\frac{1}{100}}{3\big(3-\frac{1}{100}\big)} = \frac{1}{3\cdot 299} = \frac{1}{897} \approx 0.001115, \\ dy &= -\frac{-\frac{1}{100}}{3^2} = \frac{1}{900} \approx 0.001111.\end{aligned}$$

EMEA, 12.3.7 (= MA I, 12.2.3)

Let $F(x,y,z) = x^3 + y^3 + z^3 - 3z$. Then

$$z'_x = -\frac{F'_1(x,y,z)}{F'_3(x,y,z)} = -\frac{3x^2}{3z^2-3} = \frac{x^2}{1-z^2}\,.$$

Similarly (or by symmetry), $z'_y = \dfrac{y^2}{1-z^2}$. It follows that

$$\begin{aligned}z''_{xy} = \frac{\partial}{\partial y}\left(\frac{x^2}{1-z^2}\right) &= -\frac{x^2}{(1-z^2)^2}\cdot\frac{\partial}{\partial y}(1-z^2) = -\frac{x^2}{(1-z^2)^2}(-2z)z'_y \\ &= -\frac{x^2}{(1-z^2)^2}(-2z)\frac{y^2}{1-z^2} = \frac{2x^2y^2z}{(1-z^2)^3}\,.\end{aligned}$$

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EMEA, 12.3.11 (= MA I, 12.2.4)

(a) $F(1,3) = 1e^{3-3} + 1 \cdot 3^2 - 2 \cdot 3 = 1 + 9 - 6 = 4$, so the point $(1,3)$ does lie on the level curve $F(x,y) = 4$.

The first-order partial derivatives of F are

$$F_1'(x,y) = e^{y-3} + y^2 \quad \text{and} \quad F_2'(x,y) = xe^{y-3} + 2xy - 2.$$

Hence, the slope of the tangent to the level curve $F(x,y) = 4$ at the point $(1,3)$ is

$$y' = -\frac{F_1'(1,3)}{F_2'(1,3)} = -\frac{1+3^2}{1+6-2} = -2.$$

The tangent therefore has the equation $y - 3 = (-2)(x-1)$, that is,

$$y = -2x + 5.$$

(b) Taking the logarithm of both sides, we get

$$(1 + c\ln y)\ln y = \ln A + \alpha \ln K + \beta \ln L.$$

Differentiation with respect to K gives

$$\frac{c}{y}\frac{\partial y}{\partial K}\ln y + (1 + c\ln y)\frac{1}{y}\frac{\partial y}{\partial K} = \frac{\alpha}{K}.$$

If we solve this equation with respect to $\partial y/\partial K$, we get

$$\frac{\partial y}{\partial K} = \frac{\alpha y}{K(1 + 2c\ln y)}.$$

In a similar fashion,

$$\frac{\partial y}{\partial L} = \frac{\beta y}{L(1 + 2c\ln y)}.$$

EMEA, 12.7.5 (= MA I, 12.3.1)

(a) $f(1.02, 1.99) = 3 \cdot 1.02^2 + 1.02 \cdot 1.99 - 1.99^2 = 1.1909.$

(b) We have $f_1'(x,y) = 6x + y$, $f_2'(x,y) = x - 2y$, so $f_1'(1,2) = 8$, $f_2'(1,2) = -3$. The linear approximation formula then gives

$$f(1.02, 1.99) \approx f(1,2) + 8 \cdot 0.02 - 3 \cdot (-0.01) = 1.19.$$

The error (= exact value − approximate value) is 0.0009.

EMEA, 12.7.6 (= MA I, 12.3.2)

The linear approximation formula yields

$$\begin{aligned} v(1.01, 0.02) &\approx v(1,0) + v_1'(1,0) \cdot 0.01 + v_2'(1,0) \cdot 0.02 \\ &= -1 - \tfrac{4}{3} \cdot 0.01 + \tfrac{1}{3} \cdot 0.02 = -1 - \tfrac{1}{3} \cdot 0.02 \approx -1.0067. \end{aligned}$$

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EMEA, 12.7.1 (= MA I, 12.3.3)

In both (a) and (b) we use formula (1) on page 442 (formula (3) on page 433 in MA I) to give the approximation

$$f(x,y) \approx f(0,0) + f_1'(0,0)x + f_2'(0,0)y.$$

(a) For $f(x,y) = \sqrt{1+x+y}$ we get

$$f_1'(x,y) = f_2'(x,y) = \frac{1}{2\sqrt{1+x+y}},$$

so the linear approximation to $f(x,y)$ about $(0,0)$ is

$$f(x,y) \approx 1 + \tfrac{1}{2}x + \tfrac{1}{2}y.$$

(b) For $f(x,y) = e^x \ln(1+y)$,

$$f_1'(x,y) = e^x \ln(1+y) \qquad \text{and} \qquad f_2'(x,y) = \frac{e^x}{1+y}.$$

Here, $f(0,0) = f_1'(0,0) = e^0 \ln 1 = 0$ and $f_2'(0,0) = 1$. That yields

$$f(x,y) = e^x \ln(1+y) \approx 0 + 0 \cdot x + 1 \cdot y = y.$$

EMEA, 12.7.7 (= MA I, 12.3.6)

We shall use formula (3) on page 444 (formula (4) on page 434 in MA I) to find an equation for the tangent plane.

(a) Here, $\partial z/\partial x = 2x$ and $\partial z/\partial y = 2y$. At the point $(1,2,5)$, we get $\partial z/\partial y = 2$ og $\partial z/\partial x = 4$, so the tangent plane at this point has the equation

$$z - 5 = 2(x-1) + 4(y-5) \iff z = 2x + 4y - 5.$$

(b) From $z = (y-x^2)(y-2x^2) = y^2 - 3x^2y + 2x^4$ we get $\partial z/\partial x = -6xy + 8x^3$ and $\partial z/\partial y = 2y - 3x^2$. Thus, at $(1,3,2)$ we have $\partial z/\partial x = -10$ and $\partial z/\partial y = 3$. The tangent plane is given by the equation

$$z - 2 = -10(x-1) + 3(y-3) \iff z = -10x + 3y + 3.$$

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Exam problem 38

(a) Computing differentials, we get

$$\begin{aligned} v\,d(u^2) + u^2\,dv - du &= 3x^2\,dx + 6y^2\,dy \\ e^{ux}\,d(ux) &= y\,dv + v\,dy, \end{aligned}$$

that is,

$$\begin{aligned} 2uv\,du + u^2\,dv - du &= 3x^2\,dx + 6y^2\,dy \\ ue^{ux}\,dx + xe^{ux}\,du &= y\,dv + v\,dy. \end{aligned}$$

If we substitute the values $x = 0$, $y = 1$, $u = 2$, and $v = 1$, we get

$$\begin{aligned} 4\,du + 4\,dv - du &= 6\,dy \\ 2\,dx + 0\,du &= dv + dy. \end{aligned}$$

After a bit of calculation this yields

$$du = -\frac{8}{3}\,dx + \frac{10}{3}\,dy \qquad \text{and} \qquad dv = 2\,dx - dy$$

at the point P. Hence, at this point

$$\frac{\partial u}{\partial y} = \frac{10}{3} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 2.$$

(b) We get

$$\Delta u \approx du = -\frac{8}{3}\,dx + \frac{10}{3}\,dy = -\frac{8}{3}\cdot 0.1 + \frac{10}{3}\cdot(-0.2) = -\frac{2.8}{3} \approx -0.933$$

and

$$\Delta v \approx dv = 2\,dx - dy = 2\cdot 0.1 - (-0.2) = 0.4.$$

Exam problem 57

(a) Since

$$f(tx, ty) = (ty)^3 + 3(tx)^2(ty) = t^3y^3 + 3t^3x^2y = t^3 f(x, y),$$

f is homogeneous of degree 3. It follows that the desired constant is $k = 3$, cf. Euler's theorem.

Of course, we could also calculate directly:

$$xf_1'(x, y) + yf_2'(x, y) = x\cdot 6xy + y(3y^2 + 3x^2) = 3y^3 + 9x^2y = 3f(x, y).$$

(b) For every value of x, the function $F(x, y) = y^3 + 3x^2y$ is strictly increasing with respect to y, with $F(x, y) \to -\infty$ as $y \to -\infty$ and $F(x, y) \to \infty$ as $y \to \infty$. It follows that the equation $F(x, y) = -13$ defines y as a function of x over the entire real line.

Implicit differentiation gives

$$3y^2y' + 6xy + 3x^2y' = 0,$$
$$y' = -\frac{6xy}{3x^2+3y^2} = -\frac{2xy}{x^2+y^2}\,.$$

This is the slope of the tangent to the curve at the point (x, y). With $(x, y) = (2, -1)$ we get $y' = 4/5$. Hence, the tangent to the curve at the point $(2, -1)$ is given by the equation

$$y - (-1) = \frac{4}{5}(x-2), \qquad \text{that is,} \qquad y = \frac{4}{5}x - \frac{13}{5}\,.$$

(c) From $y' = \dfrac{-2xy}{x^2+y^2}$ we get

$$y'' = \frac{(-2y - 2xy')(x^2+y^2) - (-2xy)(2x + 2yy')}{(x^2+y^2)^2}\,.$$

At the point $(2, -1)$ we have $y' = 4/5$ and

$$y'' = \frac{\left(2 - \frac{16}{5}\right)(4+1) - 4\left(4 - \frac{8}{5}\right)}{(4+1)^2} = \frac{(10-16) - \left(16 - \frac{32}{5}\right)}{25} = -\frac{78}{125} < 0.$$

Thus, y is a *concave* function of x around this point.

(d) Since $y(y^2 + 3x^2) = -13$, we have $(x, y) \neq (0, 0)$, and therefore

$$y = -\frac{13}{y^2+3x^2} < 0.$$

This shows that all points on the curve lie below the x-axis.

In part (b) we showed that

$$y' = -\frac{2xy}{x^2+y^2} = \left(\frac{-2y}{x^2+y^2}\right)\cdot x.$$

Since $\dfrac{-2y}{x^2+y^2} > 0$, we have

$$y' < 0 \ \text{ for } \ x < 0 \quad \text{and} \quad y' > 0 \ \text{ for } \ x > 0.$$

This means that y decreases when x increases in $(-\infty, 0]$, and increases when x increases in $[0, \infty)$. Hence, y attains its least value, $y_{\min}$, for $x = 0$, and so $(y_{\min})^3 + 0 = -13$, which yields $y_{\min} = \sqrt[3]{-13} = -\sqrt[3]{13}\,$.

(Alternatively we could try to solve the problem

$$\text{minimize } y \quad \text{subject to} \quad y^3 + 3x^2y = -13 \qquad (*)$$

by Lagrange's method. The Lagrangian is

$$\mathcal{L} = y - \lambda(y^3 + 3x^2y + 13),$$

and the equations $\mathcal{L}'_x = \mathcal{L}'_y = 0$ give

$$-6\lambda xy = 0, \tag{1}$$
$$1 - 3\lambda y^2 - 3\lambda x^2 = 0. \tag{2}$$

We can see from equation (2) that we must have $\lambda \neq 0$. Moreover, we showed above that $y < 0$. Hence, from (1) we get $x = 0$, and the constraint yields $y = -\sqrt[3]{13}$. This is the only *possible* solution of the problem $(*)$. But it then remains to show that it really *is* a solution of the problem.)

Exam problem 63

(a) Implicit differentiation with respect to x in the equation $3xe^{xy^2} - 2y = 3x^2 + y^2$ gives

$$3e^{xy^2} + 3xe^{xy^2}(y^2 + 2xyy') - 2y' = 6x + 2yy'.$$

With $x = 1$ and $y = 0$, we get

$$3 - 2y'(1) = 6, \quad \text{which gives} \quad y'(1) = -3/2.$$

Hence, the slope of the graph at the point $(x^*, y^*) = (1, 0)$ is $3/2$.

The linear approximation to $y(x)$ about this point is therefore

$$y(x) \approx y(1) + y'(1)(x-1) = 0 + (-\frac{3}{2})(x-1) = -\frac{3}{2}x + \frac{3}{2}.$$

(b) We want to study the model

$$pF'(L) - r = 0 \tag{1}$$
$$pF(L) - rL - B = 0 \tag{2}$$

Differentiating the equations, and isolating the terms in dp and dL on the left-hand side, we get

$$F'(L)\,dp + pF''(L)\,dL = dr \tag{3}$$
$$F(L)\,dp + (pF'(L) - r)\,dL = L\,dr + dB \tag{4}$$

Since $pF'(L) = r$, equation (4) can be simplified to

$$F(L)\,dp = L\,dr + dB \tag{4'}$$

Hence,

$$dp = \frac{L}{F(L)}\,dr + \frac{1}{F(L)}\,dB,$$

and

$$dL = \frac{1}{pF''(L)}(dr - F'(L)\,dp) = \frac{F(L) - LF'(L)}{pF''(L)F(L)}\,dr - \frac{F'(L)}{pF''(L)F(L)}\,dB.$$

It follows that

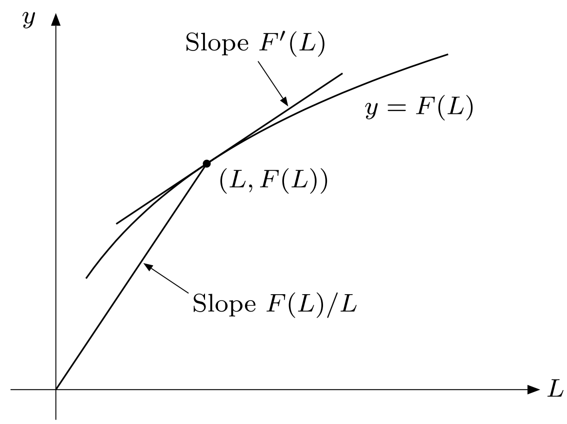
$$\frac{\partial p}{\partial r} = \frac{L}{F(L)}, \quad \frac{\partial p}{\partial B} = \frac{1}{F(L)}, \quad \frac{\partial L}{\partial r} = \frac{F(L) - LF'(L)}{pF(L)F''(L)}, \quad \frac{\partial L}{\partial B} = -\frac{F'(L)}{pF(L)F''(L)}.$$

(c) We know that $p > 0$, $F'(L) > 0$, and $F''(L) < 0$. Also, by equation (2), $F(L) = (rL + B)/p > 0$. Hence, it is clear that $\underline{\partial p/\partial r > 0}$, $\underline{\partial p/\partial B > 0}$, and $\underline{\partial L/\partial B > 0}$.

To find the sign of $\partial L/\partial r$, we need the sign of $F(L) - LF'(L)$. From equations (1) and (2), we get $F'(L) = r/p$ and $F(L) = (rL + B)/p$, so

$$F(L) - LF'(L) = B/p > 0.$$

Therefore $\underline{\partial L/\partial r < 0}$.



Exam problem 98 (c)

The figure shows the geometrical meaning of the inequality $F(L) > LF'(L) \iff F(L)/L > F'(L)$.

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Extra problems:

EMEA, 12.7.3 (= MA I, 12.3.5)

We calculate the partial derivatives of g^*:

$$\begin{aligned}\frac{\partial g^*}{\partial \mu} &= \frac{1}{1-\beta}\big((1+\mu)(1+\varepsilon)^\alpha\big)^{1/(1-\beta)-1}(1+\varepsilon)^\alpha \\ &= \frac{1}{1-\beta}\big((1+\mu)(1+\varepsilon)^\alpha\big)^{\beta/(1-\beta)}(1+\varepsilon)^\alpha, \\ \frac{\partial g^*}{\partial \varepsilon} &= \cdots = \frac{1}{1-\beta}\big((1+\mu)(1+\varepsilon)^\alpha\big)^{\beta/(1-\beta)}(1+\mu)\alpha(1+\varepsilon)^{\alpha-1}.\end{aligned}$$

(Among other things, we have used that $\dfrac{1}{1-\beta} - 1 = \dfrac{\beta}{\beta - 1}$.) Hence, at the point $(\mu, \varepsilon) = (0,0)$, we have $\partial g^*/\partial\mu = 1/(1-\beta)$ and $\partial g^*/\partial\varepsilon = \alpha/(1-\beta)$, and so for small values of μ and ε, we have

$$g^*(\mu,\varepsilon) \approx g^*(0,0) + \frac{1}{1-\beta}\cdot\mu + \frac{\alpha}{1-\beta}\cdot\varepsilon = \frac{1}{1-\beta}(\mu + \alpha\varepsilon),$$

because $g^*(0,0) = 0$.

Exam problem 89

(a) We differentiate the equation with respect to x (keeping y constant). That yields

$$3z^2\ln z + z^3\frac{1}{z} = 6z^2 z_1'(x,y)\ln z + 2z^3\frac{1}{z}z_1'(x,y),$$

that is,

$$3z^2\ln z + z^2 = (6z^2\ln z + 2z^2)z_1'(x,y) \qquad (*)$$

Putting $x = y = z = e$, we get

$$3e^2 + e^2 = (6e^2 + 2e^2)z_1'(e,e),$$

and so

$$z_1'(e,e) = \frac{4e^2}{8e^2} = \frac{1}{2}\,.$$

If we differentiate $(*)$ with respect to x again, we get

$$\begin{aligned}&6z\ln z + 3z + 2z \\ &\qquad = (12zz_1'\ln z + 6z^2\frac{1}{z}z_1' + 4zz_1')z_1' + (6z^2\ln z + 2z^2)z_{11}'' \\ &\qquad = (12z\ln z + 6z + 4z)(z_1')^2 + (6z^2\ln z + 2z^2)z_{11}''.\end{aligned}$$

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Then, with $x = y = z = e$ and $z_1'(e,e) = 1/2$ in this equation, we get

$$\begin{aligned} 6e + 3e + 2e &= (12e + 6e + 4e)\frac{1}{4} + (6e^2 + 2e^2)z_{11}''(e,e), \\ 11e &= \frac{11e}{2} + 8e^2 z_{11}''(e,e), \end{aligned}$$

which finally gives

$$z_{11}''(e,e) = \frac{11}{16e}\,.$$

(b) Differentiation with respect to x gives

$$3x^2F(xy) + x^3F'(xy)(y + xy') + e^{xy}(y + xy') = 1.$$

Note that it says $F(xy)$, not $F(x,y)$, in the problem. (F is a function of one variable, not two.) And by the rule for differentiation of products, $(xy)' = y + xy'$.

Then, with $x = 1$, $y = 0$, and $F(0) = 0$, we get

$$3F(0) + F'(0)(0 + y)' + e^0(0 + y') = 1 \iff F'(0)y' + y' = 1,$$

that is,

$$y' = \frac{1}{F'(0) + 1}\,.$$

Exam problem 92

(a) Let $F(x,y) = xe^{x^2y} + 3x^2 - 2y - 4$. The formula for implicit differentiation then yields

$$\frac{dy}{dx} = -\frac{F_1'(x,y)}{F_2'(x,y)} = -\frac{e^{x^2y} + 2x^2ye^{x^2y} + 6x}{x^3e^{x^2y} - 2}\,.$$

In particular, at the point $(x,y) = (1,0)$ we get

$$\frac{dy}{dx} = -\frac{e^0 + 0 + 6}{e^0 - 2} = 7.$$

(b) Differentiation with respect to z gives

$$\begin{aligned} \frac{dx}{dz}e^y + xe^y\frac{dy}{dz} + \frac{dy}{dz}f(z) + yf'(z) &= 0 \\ \frac{dx}{dz}g(x,y) + x\Big(g_1'(x,y)\frac{dx}{dz} + g_2'(x,y)\frac{dy}{dz}\Big) + 2z &= 0 \end{aligned}$$

By rearranging these equations we get

$$\begin{aligned} e^y\frac{dx}{dz} + \big(xe^y + f(z)\big)\frac{dy}{dz} &= -yf'(z) \\ \big(g(x,y) + xg_1'(x,y)\big)\frac{dx}{dz} + \qquad xg_2'(x,y)\frac{dy}{dz} &= -2z \end{aligned}$$

This can be viewed as a linear equation system with dx/dz and dy/dz as the unknowns. The determinant of the system is

$$
\begin{aligned}
D &= \begin{vmatrix} e^y & xe^y + f(z) \\ g(x,y) + xg_1'(x,y) & xg_2'(x,y) \end{vmatrix} \\
&= xe^y\big(g_2'(x,y) - g(x,y) - xg_1'(x,y)\big) - f(z)\big(g(x,y) + xg_1'(x,y)\big),
\end{aligned}
$$

and by Cramer's rule,

$$
\frac{dx}{dz} = \frac{1}{D}\begin{vmatrix} -yf'(z) & xe^y + f(z) \\ -2z & xg_2'(x,y) \end{vmatrix} = \frac{1}{D}\big[-xyf'(z)g_2'(x,y) + 2z\big(xe^y + f(z)\big)\big]
$$

and

$$
\begin{aligned}
\frac{dy}{dz} &= \frac{1}{D}\begin{vmatrix} e^y & -yf'(z) \\ g(x,y) + xg_1'(x,y) & -2z \end{vmatrix} \\
&= \frac{1}{D}\big[-2e^y z + y\big(g(x,y) + xg_1'(x,y)\big)f'(z)\big].
\end{aligned}
$$

(Of course, instead of taking derivatives directly, we could have computed differentials, but that would lead to almost exactly the same calculations.)

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