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On the solution to the first compulsory term paper in ECON3120/4120 Mathematics 2, spring 2008

Problem 1 (Each letter weighted 10%.)

(a) We differentiate twice – it might simplify to write $\ln \sqrt{z} = \frac{1}{2} \ln z$, although it is by no means essential:

$$
\begin{aligned}
f'(x) &= \frac{x}{1+x^2} + e^x - x^{-x} \\
f''(x) &= \frac{(1+x^2) - x \cdot 2x}{(1+x^2)^2} + e^x + e^{-x} \\
&= \frac{\frac{2}{1+x^2} - 1}{1+x^2} + e^x + e^{-x}
\end{aligned}
$$

where we used the hint in the last line. Now the only troublesome term is the «-1» – hadn't it been for that, we would have had «everything positive». There are many ways to proceed from here, for example to re-write into:

$$f''(x) = \frac{2}{(1+x^2)^2} - \frac{1}{1+x^2} + e^x + e^{-x}$$

and notice that $\frac{1}{1+x^2} \le 1$ so that

$$f''(x) \ge \frac{2}{(1+x^2)^2} - 1 + e^x + e^{-x}.$$

Now since $x \ge 0$ we have $-1 + e^x \ge 0$ (those of you who forgot the restriction $x \ge 0$ would nevertheless notice that $e^x + e^{-x} \ge 2$ always), so $f'' > 0$ and hence f is convex, Q.E.D.

(b) We are given that $f'(0) = 0$, and by convexity, f is nondecreasing[1]. It is easy to see that f is not constant near 0 (and it also follows from the fact that $f'' > 0$ from (a), but you don't need to have completed (a) to see this), so therefore f is in fact *strictly* increasing. Hence it has *at most* one zero. It is also easy to see that $f \to \infty$ as $x \to \infty$, so by the intermediate value theorem («skjæringssetningen»), the only way f can have *no* zeroes is when $f(0) > 0$. We check $f(0)$:

$$f(0) = \ln \sqrt{1} + e^0 + e^0 - K = 2 - K.$$

So f has one zero when $K \ge 2$ and no zeroes otherwise.

Note: A very common (applies to a vast majority of you!) sin of omission, was failing to argue for existence. Even if $f(0) < 0$ and f is increasing, it does not necessarily follow that f has a zero; to complete the argument, you might either check that $f \to \infty$, or

[1] Those of you who forgot that f is only defined for $x \ge 0$ – that is, you took f to be defined everywhere – will get a slight modification of the reasoning. Not much score reduced though.

point out that *both* the domain is $[0, \infty)$ and f is strictly convex. (The latter fact is of course not sufficient if the domain is bounded – you may eliminate any zero by cutting off the domain slightly to the left, if $f(0) < 0$.) You might think that this is too trivial to mention, but honestly I refuse to believe that everyone had a clear idea on the domain of definition and its significance.

(c) By the same reasoning as in (b), we have that f is strictly increasing. Therefore it is one-to-one («en-entydig») and has an inverse g, Q.E.D.
We also know that if $y = f(x)$, then $g'(y) = 1/f'(x)$ (look up or deduce). So we need to find out for which x we have $y = 2 - K$ (or more precisely, $y \to (2-K)^+$, see below). If you did solve (b), you found that this is $x = 0$. However, it can be found without attempting to solve (b): since we want to show that $g' \to +\infty$, we want an x value for which $f'(x) = 0$, and it is given information that this holds for $x = 0$. Therefore you should – irrespectively of whether you solved (b) – be able to get the idea of checking $x = 0$, which of course will lead you to $y = 2 - K$.

We are almost done – the statement we want to prove involves a one-sided limit $y \to (2-K)^+$, and a *positive* infinite result. Now g is defined on the set of values f can obtain, which is $[2-K, \infty)$, so the appropriate limit is to approach $(2-K)$ from above. Then $f'(x)$ approaches zero from above, so $1/f'(x)$ approaches $+\infty$. We are done.

Notes:

- Too many of you forgot to show that there is an inverse, and lost half the score (maybe a bit harsh, I know).
- Too many of you omitted the argument why the limit is *positive* infinity.

Problem 2 (Each letter weighted 6%.)

(a) The condition that $q > 0$ was to help those of you who argue like this:

$$\lim_{x\to\infty}(x^p - x^q) = \lim_{x\to\infty}(x^q(x^{p-q} - 1)) = \underline{\underline{\pm\infty}}$$

(since both $q > 0$ and $p - q > 0$, both factors tend to $+\infty$). However, we do obviously not need $q > 0$ as long as $p > \max\{0, q\}$; if $q \le 0$ then x^q is bounded. A unified argument which does not use the sign of q, is

$$\lim_{x\to\infty}(x^p - x^q) = \lim_{x\to\infty}(x^p(1 - x^{q-p}))$$

where the first factor tends to $+\infty$ while the latter tends to 1.

(b) First we expand: $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ (hint: you might check your calculations by (1) noticing that the highest order coefficient matches, and (2) that the three zeroes are the same). Then we perform long division, from which we get (after some calculation)

$$\frac{x^6 - 6x^5 + 11x^4}{x^3 - 6x^2 + 11x - 6} = x^3 + 6 + 6\frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)}$$

where in the latter term we have used the calculated identity $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ «in reverse».

Now re-write the using partial fractions:

$$\frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3} \tag{*}$$
$$6x^2 - 11x + 6 = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2)$$

Recall that this is an equality between *functions* (i.e. to hold for *all* x), not an equation in x. You may either expand the right hand side and equate coefficients, or – if you know what you are doing (and many of you don't, see the Notes below) – observe that both sides are quadratic forms an can be matched by inserting x values to obtain equations that A, B and C have to obey. The easiest x values to insert would be $x = 1$, $x = 2$, $x = 3$ (each eliminating all but one of the right hand side terms). In any method, we get

$$A_1 = \frac{1}{2}, \qquad A_2 = -8, \qquad A_3 = \frac{27}{2}.$$

So $\frac{x^6-6x^5+11x^4}{x^3-6x^2+11x-6} = x^3 + 6 + \left(\frac{3}{x-1} - \frac{48}{x-2} + \frac{81}{x-3}\right)$ and we are (finally!) ready to *start* integrating. Fortunately that is easy, and the general antiderivative is (remember the absolute value signs and the C!):

$$\underline{\underline{\tfrac{1}{4}x^4 + 6x + 3\ln|x-1| - 48\ln|x-2| + 81\ln|x-3| + C.}}$$

Notes:

- Too many mistakes – conceptual, not typos – in your long division.
- You *have to* carry on with division until the degree of the numerator is *strictly smaller!* Many of you got $6x^3$ in the numerator, and equated this 3rd degree function with the second degree expression on the right hand side of (*). A quadratic can of course never be forced to match an arbitrary cubic, but inserting three values will not disclose the error; here the short cut prevents you from discovering that you had made a mistake.

(c) The natural substitution is $u = \ln\ln x$ – well since virtually none of you pointed out already here that the $\ln$ is positive on the scope of integration, it should more accurately be $\ln|\ln x|$ – giving $du = \frac{1}{\ln x}\frac{1}{x}dx$ and the indefinite integral $\int \frac{du}{u} = \ln|u| + C = \ln|\ln|\ln x|| + C$. To evaluate the *definite* integral, we observe that $\ln(e^e + 1) > \ln e^e$, and taking $\ln$ of both sides once more we still get something positive, so the absolute value signs may be dropped, and we get:

$$\ln\ln\ln(e^e + 1) - \ln\ln\ln(e^e) = \underline{\underline{\ln\ln\ln(e^e + 1)}}.$$

Notes:

- *Either* you first calculate the indefinite integral (including substituting back, and then write e.g. as above – *or* you substitute for the limits of integration also. (You *may* express the limits as e.g. $u(1 + e^e)$, but do not write x-limits when you are supposed to have u-limits.)
- Arguably a slightly more clever substitution is $v = \ln|\ln|\ln x||$. Try it!

(d) The integrand is an odd function (namely, one for which $f(-x) = -f(x)$. Since the scope of integration is a symmetric set around 0, the «area under the graph for $x > 0$» matches the «area over the graph for $x < 0$». So the integral is zero.

Notes:

- Strictly speaking, you should also point out that the integral is proper.
- For a formal proof, split the integral at zero, and substitute $y = -x$ in one of them. Try it!

(e) We have $f(y) = \displaystyle\int_0^\infty t^y e^{-t}\,dt$, defined for $y \geq 0$. Note that there is an improper integral, defined to be the limit of $\int_0^R$ as $R \to \infty$ (when the limit exists!)

To prove that $f(y+1) = (y+1)f(y)$, we integrate by parts, using $u = t^{y+1}$ and $v' = e^{-t}$:

$$\begin{aligned} f(y+1) &= \int_0^\infty t^{y+1} e^{-t}\,dt \\ &= \lim_{R\to\infty} \int_0^R t^{y+1} e^{-t}\,dt \\ &= \lim_{R\to\infty} \Big(\big[t^{y+1}e^{-t}\big]_{t=0}^{t=R} - \int_0^R (y+1)\cdot t^{y+1-1}\cdot(-e^{-t})\,dt \Big). \end{aligned}$$

The limit of the latter integral is $(y+1)f(y)$ after taking into account the negative sign(s), so the proof will be complete if we can prove $\lim\limits_{R\to\infty} \big[t^{y+1}e^{-t}\big]_{t=0}^{t=R} = 0$. Since $0^{y+1}e^{-0} = 0$, we only need to show $\lim\limits_{R\to\infty} R^{y+1}e^{-R} = 0$. This is the important limit in EMEA p. 265 (MA I) page 224, and we are done.

Note: *Please* be more precise about the usage of limits here. An exam committee could easily punish such quite harsher than what I did. The «≈ 6» points means, roughly that it is at least a marginal A.

Problem 3 For notes for both (a) and (b), see at the end of part (b).

(a) A stationary point for x is when $\dot{x} = 0$, which holds for $\underline{\underline{t = K}}$.

(b) Since $x > 1$ we have $x/\ln x \neq 0$ and hence no constant solution. We separate the differential equation into

$$\frac{\ln x}{x} dx = (t - K)dt$$

and integrate (for the left hand side, substituting $u = \ln x$) into

$$\tfrac{1}{2}(\ln x)^2 = \tfrac{1}{2}t^2 - tK + C.$$

We are only interested in the C for which $x(K) = e$:

$$\tfrac{1}{2}1^2 = \tfrac{1}{2}K^2 - K^2 + C.$$

Inserting for $C = \frac{1}{2}(1+K^2)$ we get

$$(\ln x)^2 = t^2 - 2tK + K^2 + 1$$

or

$$x(t) = \underline{\underline{\exp\sqrt{(t-K)^2+1}}}$$

Notes:

- Part (a) was weighted 10 of the 25 points, which is maybe all too much considering its simplicity (a lot of you seem to have forgotten all about it though). For part (b), I awarded 5 for the the method (separating and at least attempting to integrate), 5 for general solution, 5 for particular solution.
- Points were awarded irrespectively of whether the answer was given under a «(a)» or «(b)» headline, for example if the stationary point was found by way of the general solution (cumbersome, but legal).
  - A lot of you tried to find a zero for the radicand after solving. That is incorrect. It would of course be OK if you found en *extremum* for the radicand (since $h \mapsto e^{\sqrt{h}}$ is a strictly increasing function, then an extremum t^* for $h(t)$ would be a stationary point for x), which is the axis of symmetry for the quadratic $\frac{1}{2}t^2 - tK + C$ – that is, $t = -(-2K/2) = K$.
- Lots of strange mistakes – like confusing K with the constant of integration (I dare not think what could have been the result had I called K «C» instead ...), forgetting the C altogether, adding it to the exponential at the very end, ...

Problem 4 (Each letter weighted 5%.)

(a) <u>True.</u> Element (i,j) of $\mathbf{A}+\mathbf{B}$ is $a_{ij}+b_{ij}$, becomes element (j,i) after transposition. On the other hand, element (i,j) of each matrix becomes element (j,i) after transposition, and then add them. (Or see the book.)

(b) <u>False.</u> Calculating the difference we are left with $\mathbf{AB}-\mathbf{BA}$ which is zero only in special cases.

(c) <u>True.</u> *One* nonzero price does in fact suffice; take for example price number i. Then

$$p_i(x_i - a_i) = \sum_{\text{all } j \text{ except } i} p_j(x_j - a_j)$$
$$x_i = a_i + \frac{1}{p_i}\sum_{\text{all } j \text{ except } i} p_j(x_j - a_j)$$

which shows that (1) all the other x_j may be chosen freely, and (2) when they are chosen, then x_i is determined. Ergo, $n-1$ degrees of freedom.

Note: A mathematician (like myself) will argue that you should point out (and it is much simpler with a mathematical argument – if you understand what happens!) that we actually need $\mathbf{p} \neq \mathbf{0}$ – if all prices were zero, we would have n degrees of freedom (you can take whatever you want of anything!) and the proposition would be false. Then on the other hand, you are economists and probably not very interested in a case where *everything* is for free, are you? This maybe makes this problem a bit too rough for an exam, and there was a slight discrepancy between the graders on this point too. (We did not attempt to judge this point uniformly.)