On term paper #2, ECON3120/4120 Mathematics 2, spring 2008

Problem 1 Cofactor expansion along the last row gives

$$\begin{aligned}|\mathbf{A}_u| &= -u\begin{vmatrix}1 & 1-u\\ u-1 & 3u-1\end{vmatrix} + 2u\begin{vmatrix}1 & 2u-1\\ u-1 & 1\end{vmatrix}\\ &= u\left(1-3u-u^2+2u-1+2-2(2u-1)(u-1)\right)\\ &= u\left(-u^2-u-4u^2+6u\right)\\ &= 5u^2(1-u).\end{aligned}$$

So – regardless of the right hand side (and thus, regardless of k) – there is a unique solution whenever $u \notin \{0,1\}$. The cases $u=0$ and $u=1$ have to be treated separately.

The case $u=0$ gives

$$\begin{aligned} x - \ y + \ z &= 0\\ -x + \ y - \ z &= k\\ 0x + 0y + 0z &= 0\end{aligned}$$

or – adding the first line to the second (or merely comparing those two!)

$$\begin{aligned} x - \ y + \ z &= 0\\ 0 &= k\end{aligned}$$

So for $u=0$ there is *no solution for* $k \neq 0$, while for $k=0$ there are *two degrees of freedom.*

The case $u=1$ gives

$$\begin{aligned} x + y + \ z &= 1\\ y - 2z &= k\\ y - 2z &= k\end{aligned}$$

where the last two lines are the same, so one of them can be dropped. It is easy to see that – regardless of k – we have *one degree of freedom*; if we choose $z=t$, the equation system becomes

$$\begin{aligned} x + y &= 1-2t\\ y &= k+2t\end{aligned}$$

which – for each t – has a unique solution for x and y. To summarize:

- No solution when $u = 0 \neq k$
- Two degrees of freedom when $u = k = 0$
- One degree of freedom when $u = 1$
- Unique solution otherwise.

Notes:

- You were not asked to actually find the solution. Read the problem text!
- Nevertheless, you may perform Gaussian elimination as a tool. However, whenever you divide by something, ensure that it is nonzero. If you divide by u, then the case $u = 0$ will not not be covered by your calculations, and will have to be treated separately.
- Cramer's rule does not work unless $|\mathbf{A}_u| \neq 0$. Example why: Consider the equations $ux = u$, $uy = u$, $uz = u$. Clearly, this system has the unique solution $x = y = z = 1$ whenever $u \neq 0$, while for $u = 0$ all variables are free. Now try to use Cramer's rule and see what happens.
- The problem set says «in the unknowns x, y and z». That is, k and u are *not* unknowns.
- A «zero line» does *not* imply infinitely many solutions, you have to check the rest of the equation system for consistency – which is also necessary in order to find the number of degrees of freedom.

Problem 2 We are given (for $t > 0$, $x > 0$):

$$V(t, x) = g(t)h(x)e^{-rt} - x$$

(a) The first-order conditions are

$$0 = V'_t(t^*, x^*) = h(x^*) \cdot \big(g'(t^*)e^{-rt^*} - rg(t^*)e^{-rt^*}\big)$$
$$0 = V'_x(t^*, x^*) = g(t^*)e^{-rt^*}h'(x^*) - 1$$

or:

$$\underline{\underline{g'(t^*) = rg(t^*)}}$$

$$\underline{\underline{h'(x^*) = e^{rt^*}/g(t^*)}}$$

(the latter valid because $g > 0$.)

(b) We are given that $V''_{tx}(t^*, x^*) = 0$, so the second-order test will be satisfied if $V''_{tt}(t^*, x^*) < 0$ and $V''_{xx}(t^*, x^*) < 0$.
Consider first the latter: $V''_{xx}(t, x) = g(t)e^{-rt}h''(x)$, and since $g(t) > 0$, this has the same sign as h''. So if $h''(x^*) < 0$, then $V''_{xx}(t^*, x^*) < 0$.
For V''_{tt}, we have

$$
\begin{aligned}
V''_{tt}(t,x) &= h(x)\cdot \tfrac{d}{dt}\left[(g'(t)-rg(t))e^{-rt}\right] \\
&= h(x)\cdot\left[(g''(t)-rg'(t))e^{-rt} - r(g'(t)-rg(t))e^{-rt}\right] \\
&= h(x)\cdot\left[g''(t)-rg'(t)-r(g'(t)-rg(t))\right]e^{-rt}
\end{aligned}
$$

which – since $he^{-rt} > 0$ – has the same sign as the expression in the brackets. Now the proposition we are asked to show does not involve g', so we use the hint and substitute $g'(t^*) = rg(t^*)$ from the first-order condition. Then at the stationary point we get that $0 > V''_{tt}(t^*, x^*)$ if

$$
\begin{aligned}
0 &> g''(t^*) - rg'(t^*) - r(g'(t^*) - rg(t^*)) \\
&= g''(t^*) - r\cdot rg(t^*) - 0
\end{aligned}
$$

so that the second-order condition will hold if

$$
r^2 g(t^*) > g''(t^*)
$$

which is precisely what we were asked to show.

(c) We are given $g(t) = e^{\sqrt{t}}$ and $h(x) = \ln(x+1)$, which are both positive functions. We have $g'(t) = g(t)/2\sqrt{t}$ and $h'(x) = 1/(x+1)$, so the first-order conditions become

$$
\begin{aligned}
\text{For } t^*: \quad & \frac{1}{2\sqrt{t}} g(t^*) = rg(t^*) \\
\text{implying} \quad & t^* = 1/4r^2 \\
\text{For } x^*: \quad & \frac{1}{x^*+1} = e^{rt^*}e^{-\sqrt{t^*}} \\
& = e^{\frac{1}{4r}-\frac{1}{2r}} \\
& = e^{-\frac{1}{4r}} \\
\text{implying} \quad & x^* = e^{1/4r} - 1
\end{aligned}
$$

For the second-order conditions, we easily see that $h''(x) = -(x+1)^{-2}$ which is < 0 for any x (hence also for x^*), so we only need to verify $r^2 g(t^*) > g''(t^*)$. Now the second derivative is

$$
\begin{aligned}
g''(t) &= \tfrac{1}{2}g'(t)t^{-1/2} + (-\tfrac{1}{4})g(t)t^{-3/2} \\
&= \tfrac{1}{4}\left(2g(t)t^{-1/2}\cdot\tfrac{1}{2}t^{-1/2} - g(t)t^{-3/2}\right) \\
&= \tfrac{1}{4}g(t)t^{-3/2}(t^{1/2} - 1)
\end{aligned}
$$

so that the statement $r^2 g(t^*) > g''(t^*)$ is equivalent to

$$
\begin{aligned}
r^2 g(t^*) &> \tfrac{1}{4}g(t^*)\cdot(t^*)^{-3/2}\cdot((t^*)^{1/2} - 1) \\
&\Updownarrow \quad \text{(because } g > 0\text{)} \\
r^2 &> \tfrac{1}{4}(4r^2)^{3/2}\cdot(\frac{1}{2r} - 1)
\end{aligned}
$$

The right hand side is equal to $2r^3(\frac{1}{2r} - 1) = r^2 - 2r^3$, which is $< r^2$ when $2r^3 > 0$ (and $r > 0$ is given). So the second-order test implies local maximum, for all $r > 0$.

Notes: Most of you fared well on problem 2. It has been given as an exam problem, and turned out difficult there, probably because of the time frame – the problem requires a bit of work, though maybe not much sheer brilliance. Typical errors were calculation mistakes, forgetting the second derivative test in 2c, and taking $x^* = 0$ even though V is not defined there.

Problem 3

(a) To differentiate the equation system

$$\begin{aligned} x \cdot y \cdot z \cdot ue^{-u} \cdot ve^{v} &= 2 \\ 1x + 2y + 3z + 4u + 5v &= 6 \end{aligned}$$

we first observe that $\frac{d}{dw}(we^{kw}) = e^{kw} + kwe^{kw} = e^{kw}(1 + kw)$, useful for the differentiation wrt. u and v. We proceed to get the following answer:

$$\begin{cases} yzue^{-u}ve^{v}\,dx + xzue^{-u}ve^{v}\,dy + xyue^{-u}ve^{v}\,dz & \\ +xyze^{-u}(1-u)ve^{v}\,du + xyzue^{-u}e^{v}(1+v)\,dv & = 0 \\ dx + 2dy + 3dz + 4du + 5dv & = 0 \end{cases}$$

(b) First, we note that we can simplify the first equation by dividing by the common nonzero factor $e^{-u}e^{v}$ (indeed you may even divide by $xyzue^{-u}ve^{v}$, but I will not base the following on that trick). To find a general expression for v'_x, we also note that the dy and dz terms will not be interesting; however, a look at part (c) reveals that we also want u'_z (although only in the point). So let us proceed fairly generally to get

$$\underbrace{\begin{pmatrix} xyz(1-u)v & xyzu(1+v) \\ 4 & 5 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} du \\ dv \end{pmatrix} = -\begin{pmatrix} yzuv \\ 1 \end{pmatrix} dx - \begin{pmatrix} xzuv \\ 2 \end{pmatrix} dy - \begin{pmatrix} xyuv \\ 3 \end{pmatrix} dz$$

or, equivalently – since (locally near P) we have $|\mathbf{A}| = xyz(5(1-u)v - 4u(1+v)) \neq 0$ –

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \underbrace{\frac{-1}{|\mathbf{A}|} \overbrace{\begin{pmatrix} 5 & -xyzu(1+v) \\ -4 & xyz(1-u)v \end{pmatrix}}^{\text{adj } \mathbf{A}}}_{-\mathbf{A}^{-1}} \left\{ \begin{pmatrix} yzuv \\ 1 \end{pmatrix} dx + \begin{pmatrix} xzuv \\ 2 \end{pmatrix} dy + \begin{pmatrix} xyuv \\ 3 \end{pmatrix} dz \right\} \qquad (*)$$

If we perform matrix multiplication, v'_x will be the dx coefficient of the dv line:

$$\begin{aligned} v'_x &= \frac{-1}{|\mathbf{A}|}\overbrace{\begin{pmatrix}-4 & , & xyz(1-u)v\end{pmatrix}}^{\text{second («}dv\text{»-)row of adj }\mathbf{A}}\begin{pmatrix}yzuv\\1\end{pmatrix} \\ &= \frac{4yzuv - xyz(1-u)v}{5xyz(1-u)v - 4xyzu(1+v)} \\ &= \frac{v}{x}\frac{4u - x + xu}{5v - 5uv - 4u - 4uv)} \\ &= \underline{\underline{\frac{v}{x}\frac{4u - x + xu}{5v - 9uv - 4u}}} \end{aligned}$$

(c) The new point is found by keeping x and y constant, and reducing z by 0.1. That is, $dx = dy = 0$, $dz = -1/10$. Besides, we are only interested in an approximation from point $P : (x, y, z, u, v) = (2, -1, -1, 1, 1)$, not a general expression. Inserting these coordinates, the first («du-») line of (*) becomes

$$\begin{aligned} du &= \left[\overbrace{\frac{-1}{|\mathbf{A}|}\begin{pmatrix}5 & , & -xyzu(1+v)\end{pmatrix}}^{\text{first line of }-\mathbf{A}^{-1}}\left\{\begin{pmatrix}yzuv\\1\end{pmatrix}0 + \begin{pmatrix}xzuv\\2\end{pmatrix}0 + \begin{pmatrix}xyuv\\3\end{pmatrix}\left(\frac{-1}{10}\right)\right\}\right]_P \\ &= \frac{1}{16}\begin{pmatrix}5 & , & -4\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}\left(\frac{-1}{10}\right) \\ &= \frac{11}{80} \end{aligned}$$

So the approximate value for u is $u \approx 1 + \frac{11}{80} = \underline{\underline{\frac{91}{80}}}$

Notes:

- Some of you did this in an unnecessarily cumbersome fashion – not a problem if you get it right, but could take too much time on an exam.
- In (a), you were only asked to find differentials, not to solve for (du, dv). However, those of you who did that in part (a), did not have to do so in problem (b) and (c).
- (b): «General» expression means with x, y, z u, v, not with the numbers.
- (c): Quite a few of you only calculated du at P, not $u(2, -1, -1) + du$. Read the problem.

Problem 4 We are given the problem (for strictly positive A, B and C)

$$\max_{(x,y)} 4e^x + \tfrac{1}{2}Ax^2y^2 + e^{3y} \qquad \text{subject to} \qquad x^2 + By^2 \le C, \qquad x \ge 0 \qquad \text{and} \quad y \ge 0.$$

(a) The Lagrangian is (note the signs!):

$$L(x,y) = 4e^x + \tfrac{1}{2}Ax^2y^2 + e^{3y} - \lambda(x^2 + By^2) + \mu x + \nu y.$$

The Kuhn-Tucker conditions are *(corrected from version 1!)*:

$$\begin{array}{lr} 0 = 4e^x + Axy^2 - 2\lambda x + \mu & (1) \\ 0 = 3e^{3y} + Ax^2y - 2\lambda By + \nu & (2) \\ \lambda \ge 0 \quad \text{with } \lambda = 0 \text{ if } x^2 + By^2 < C & (3) \\ \mu \ge 0 \quad \text{with } \mu = 0 \text{ if } x > 0 & (4) \\ \nu \ge 0 \quad \text{with } \nu = 0 \text{ if } y > 0 & (5) \end{array}$$

(b) To prove that $x^2 + By^2 = C$ and $xy \ne 0$, we shall show a contradiction if not:

- If $x = 0$ then (1) becomes $0 = 4e^x + \mu$, which is impossible since $e^x > 0$ and $\mu \ge 0$.
- If $y = 0$ then (2) becomes $0 = 3e^{3y} + \nu$, impossible by a similar argument.
- If $x^2 + By^2 \ne C$ (that is, $< C$), then $\lambda = 0$ by (3). In the same manner as above, the right hand side of (1) will be the sum of nonnegative terms $4e^x + Axy^2 + \mu$ where the exponential is strictly positive. (There would be a similar contradiction in (2) too.) So it is impossible that $x^2 + By^2 \ne C$.

Arguably it is more striking to re-write (1), (2) into

$$2\lambda x = 4e^x + Axy^2 + \mu \qquad (1')$$
$$2\lambda By = 3e^{3y} + Ax^2y + \nu \qquad (2')$$

and observe that both right hand sides are strictly positive; hence we cannot have $xy\lambda = 0$, unifying all three bullet points in one elegant operation which one is probably unlikely to spot until long after one has completed a proof in the first place.

Notes: (a) was easy. Part (b) was plagued by weird logic:

- You cannot from (1) write $2\lambda = (4e^x + Axy^2 + \mu)/x$ without *assuming* $x \ne 0$. And assuming what you wanted to prove, destroys the entire argument – then you have to check $x = 0$ separately afterwards (which you should have done in the first place!)
- «Assume not»: you must understand that the negation of «P *and* Q *and* R» is «not P *or* not Q *or* not R» (where «or» means «and/or»). You must falsify *each* $x = 0$, $y = 0$, $x^2 + By^2 < C$. (Indeed, $x = y = 0$ *and* $x^2 + By^2 < C$ implies $(x,y) = (0,0)$.)
- Some of you seem to think that if $x = 0$ then $\mu > 0$. Wrong – they can both hold with equality (but not both with inequality! Mind the difference!)