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The problem from lecture 8. april 2008

The problem was as follows:

Find and classify the stationary points of f given by $f(x,y) = x^4 - x^2 - y^3 + y^2 + \frac{1}{2}x^2y^2$. We need the following partial derivatives:

$$f_1'(x,y) = 4x^3 - 2x + xy^2 = x(4x^2 - 2 + y^2)$$
$$f_2'(x,y) = -3y^2 + 2y + x^2y = y(-3y + 2 + x^2)$$

and also the second-order partial derivatives

$$f_{11}''(x,y) = 12x^2 - 2 + y^2$$
$$f_{22}''(x,y) = -6y + 2 + x^2$$
$$f_{12}''(x,y) = f_{21}''(x,y) = 2xy$$

so that the Hessian is

$$\begin{aligned} h(x,y) &= f_{11}''(x,y) \cdot f_{22}''(x,y) - f_{12}''(x,y) \cdot f_{21}''(x,y) \\ &= (12x^2 - 2 + y^2) \cdot (-6y + 2 + x^2) - 4x^2y^2 \end{aligned}$$

This gives four cases for stationary points, where the three first were correct at the lecture:

(I) $x = 0$ and $y = 0$, stationary point $(x_1, y_1) = \underline{\underline{(0,0)}}$

(II) $x = 0$ and $(-3y + 2 + x^2) = 0$, stationary point $(x_2, y_2) = \underline{\underline{(0, \frac{2}{3})}}$

(III) $(4x^2 - 2 + y^2) = 0$ and $y = 0$, gives two stationary points $(x_3, y_3) = \underline{\underline{(-\frac{1}{2}\sqrt{2}, 0)}}$ and $(x_4, y_4) = \underline{\underline{(\frac{1}{2}\sqrt{2}, 0)}}$

(IV) $(4x^2 - 2 + y^2) = 0$ and $(-3y + 2 + x^2) = 0$. Inserting $x^2 = 3y - 2$ into the first condition gives $12y - 10 + y^2 = 0$, which holds when $y =$

$$\begin{aligned} & y_5 = \frac{1}{2}(-12 + \sqrt{144 + 40}) = -6 + \sqrt{46} \\ \text{or} \quad & y_6 = -6 - \sqrt{46}. \end{aligned}$$

We can now find x from the relation $x^2 = 3y - 2$; however for y_6, the right hand side is negative, so there will be no x value and hence no point. For y_5 we do have $3y_5 - 2 = 3\sqrt{46} - 20 \geq 0$ and hence there are two possible x values: $x_5 = \sqrt{3y_5 - 2} = \sqrt{3\sqrt{46} - 20}$ and also $-x_5$.
This gives two stationary points $(x_5, y_5) = \underline{\underline{(\sqrt{3\sqrt{46} - 20}, -6 + \sqrt{46})}}$ and $(-x_5, y_5) = \underline{\underline{(-\sqrt{3\sqrt{46} - 20}, -6 + \sqrt{46})}}$.

To classify these using the second derivative test, we first notice that f''_{11}, f''_{22} and h only depend on x through x^2. Each of the cases give:

(I) $f''_{11}(0,0) < 0 < f''_{22}(0,0)$, so $h(0,0) < 0$ and the origin is a saddle point.

(II) $f''_{11}(0, \frac{2}{3}) = \frac{4}{9} - 2 < 0$, while
$f''_{22}(0, \frac{2}{3}) = -2 < 0$ and
$f''_{12}(0, \frac{2}{3}) = 0$ which yields $h(0, \frac{2}{3}) > 0$.
So (x_2, y_2) is a maximum point.

(III) $f''_{11}(\pm\sqrt{\frac{1}{2}}, 0) = 4 > 0$,
$f''_{22}(\pm\sqrt{\frac{1}{2}}, 0) = \frac{5}{2} > 0$, and
$f''_{12}(x, 0) = 0$ which yields $h(\pm\sqrt{\frac{1}{2}}, 0) > 0$.
So (x_3, y_3) and (x_4, y_4) are minimum points.

(IV) For $(\pm x_5, y_5)$ we may use the fact that $(\pm x_5)^2 = 3y_5 - 2$ (from the first order condition), to get rid of $\pm x_5$. We can also use the relation $y_5^2 = 12y_5 - 10$. We then get
$f''_{11}(\pm x_5, y_5) = 12(3y_5 - 2) - 2 + 12y_5 - 10 = 12(4y_5 - 3) = 12(4\sqrt{46} - 27) > 0$ (to see the positivity, observe that $4^2 \cdot 46 = 736 > 729 = 27^2$.)
$f''_{22} = -6y_5 + 2 + 3y_5 - 2 = -3y_5 < 0$ (since $y_5 = \sqrt{46} - 6 > 0$). Since these derivatives have opposite sign, we will have $h(x, y) < 0$ no matter what value of the cross-derivative, so (x_5, y_5) and $(-x_5, y_5)$ are both saddle points