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Problem solutions – voluntary term paper in ECON3120/4120 Mathematics 2, spring 2009

Problem 1

(a) Taking the natural logarithm on both sides we get

$$x \ln 3 + 3x \ln 2 = \ln 17\,,$$

which gives

$$x = \frac{\ln 17}{\ln 3 + 3\ln 2} = \frac{\ln 17}{\ln 3 + \ln 8} = \frac{\ln 17}{\ln 24} \approx 0.891\,.$$

(b) We want to solve the equation system

$$\begin{aligned} 4x^2y^2 - x^2y^4 &= 0 \\ x + xy &= 0 \end{aligned}$$

If $x = 0$, then both equations are satisfied for all values of y. Now assume $x \neq 0$. The last equation the gives $y = -1$, but if we insert this in the first equation we get $4x^2 - x^2 = 0$, i.e. $3x^2 = 0$, which gives $x = 0$. Hence we cannot have $x \neq 0$.

Conclusion: The equation system has the solutions $(x, y) = (0, a)$, where a is arbitrary.

Problem 2

(a) Introduce $u = \sqrt{2-3x}$ as a new variable. This gives $u^2 = 2 - 3x$, so $x = (2-u^2)/3$ and $dx = -(2u/3)\,du$. The integral then becomes

$$\begin{aligned} \int_{-2/3}^{1/3} \frac{x\,dx}{\sqrt{2-3x}} &= \int_2^1 \frac{2-u^2}{3u}\Big(-\frac{2}{3}u\Big)du = \int_2^1 \frac{2u^2-4}{9}\,du \\ &= \Big|_2^1 \Big(\frac{2u^3}{27} - \frac{4u}{9}\Big) = \Big(\frac{2}{27} - \frac{4}{9}\Big) - \Big(\frac{16}{27} - \frac{8}{9}\Big) = -\frac{2}{27} \end{aligned}$$

(Note that in the integral with respect to u we uses the u-values of the limits of integration.) If we choose to calculate the indefinite integral completely in terms of x before evaluating the definite integral, we get

$$\begin{aligned} \int \frac{x\,dx}{\sqrt{2-3x}} &= \int \frac{2u^2-4}{9}\,du = \frac{2u^3}{27} - \frac{4u}{9} + C \\ &= \frac{2}{27}u(u^2-6) + C = \frac{2}{27}\sqrt{2-3x}\,(-3x-4) + C\,. \end{aligned}$$

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It is also possible to use integration by parts, which in a natural way leads to

$$\int \frac{x\,dx}{\sqrt{2-3x}} = -\frac{2}{3}x(2-3x)^{1/2} - \frac{4}{27}(2-3x)^{3/2} + C\,,$$

which equals the expression we found above.

(b) Let $g(x) = \ln f(x) = \dfrac{e}{1+\ln x}\ln x = \dfrac{e\ln x}{1+\ln x} = \dfrac{e}{(1/\ln x)+1}$. Because $\ln x \to -\infty$ as $x \to 0^+$, we have $\lim\limits_{x\to 0^+} g(x) = e$. Now, $f(x) = e^{g(x)}$, and because the exponential function $u \mapsto e^u$ is continuous, we get

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{g(x)} = e^{(\lim_{x\to 0^+} g(x))} = e^e.$$

Problem 3

(a) We use Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 0 & -2 & 2 \\ 0 & 2 & -1 & -1 & 3 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} -1 \\ \big\downarrow \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 1 & 0 & -2 & 2 \\ 0 & 2 & -1 & -1 & 3 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ -1/2 \quad 1/2 \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1/2 & -3/2 & 1/2 \\ 0 & 1 & -1/2 & -1/2 & 3/2 \\ 0 & 0 & 0 & 3 & 0 \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ 1/2 \quad 1/6 \quad 1/3 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & -1/2 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This shows that we can choose arbitrary values for x_3, for instance, and the solutions are then

$$x_1 = \frac{1}{2} - \frac{1}{2}a, \quad x_2 = \frac{3}{2} + \frac{1}{2}a, \quad x_3 = a, \quad x_4 = 0,$$

where a is arbitrary.

(b) Direct calculation gives

$$\mathbf{AA}' = \begin{pmatrix} 6 & 4 & 0 \\ 4 & 6 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

If we let $\mathbf{B} = \begin{pmatrix} 17 & -12 & 4 \\ -12 & 18 & -6 \\ 4 & -6 & 20 \end{pmatrix}$, then

$$(\mathbf{AA}')\mathbf{B} = \begin{pmatrix} 54 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 54 \end{pmatrix} = 54\,\mathbf{I}_3\,,$$

and it follows that $(\mathbf{AA}')^{-1} = \frac{1}{54}\mathbf{B}$.

For *every* matrix $\mathbf{A}$ the product $\mathbf{AA}'$ is symmetric because it is equal to its own transpose: $(\mathbf{AA}')' = (\mathbf{A}')'\mathbf{A}' = \mathbf{AA}'$. Moreover, we know that if a symmetric matrix has an inverse, then the inverse matrix is also symmetric. (See formula (15.5.2)(4) and Theorem 16.6.1(c) in EMEA or formula (3.22)(4) and Theorem 6.1(c) in LA.)

Problem 4

(a) The equation $\dot{x} - 2x = k - 2t$ is a linear differential equation of order 1, and we can find the solution by using formula (9.9.5) in EMEA or (1.4.5) in MA II, with $a = -2$ and $b(t) = k - 2t$. That gives

$$x = e^{2t}\left(C + \int e^{-2t}(k-2t)\,dt\right).$$

Integration by parts yields

$$\begin{aligned}\int e^{-2t}(k-2t)\,dt &= \int (k-2t)e^{-2t}\,dt \\ &= (k-2t)\big(-\frac{1}{2}\big)e^{-2t} - \int (-2)\big(-\frac{1}{2}\big)e^{-2t}\,dt \\ &= (k-2t)\big(-\frac{1}{2}\big)e^{-2t} - \int e^{-2t}\,dt = \big(t - \frac{k}{2}\big)e^{-2t} + \frac{1}{2}e^{-2t},\end{aligned}$$

and the general solution of the differential equation is therefore

$$x = Ce^{2t} + t + \frac{1-k}{2}.$$

(b) We differentiate both sides of the equation (cf. formula (9.3.6) in EMEA or formula (10.3.9) in MA I) and get

$$\dot{x} = 2x(t) + 1 - 2t.$$

This is precisely the equation in part (a) with $k = 1$. Hence,

$$x(t) = Ce^{2t} + t$$

for a suitable value of C. Equation $(**)$ in the problem shows that we must have

$$x(0) = 2 \cdot 0 + 0 + 1 - 0 = 1.$$

It follows that $C = 1$ and $x(t) = e^{2t} + t$. A simple calculation confirms that this function does satisfy $(**)$.

Problem 5

(a) We have

$$f'(x) = \frac{4a}{1+a^2}e^{-x} - 1, \qquad f''(x) = -\frac{4a}{1+a^2}e^{-x}.$$

Because $\lim_{x\to\infty}(1-e^{-x}) = 1 - \lim_{x\to\infty} e^{-x} = 1 - 0 = 1$, we get $\lim_{x\to\infty} f(x) = -\infty$.

(b) For $x > 0$ we have $e^{-x} < 1$, and therefore $1 - e^{-x} > 0$. It follows that if $a \leq 0$, then

$$f(x) = \frac{4a}{1+a^2}(1-e^{-x}) - x \leq -x < 0 \quad \text{for all } x > 0.$$

Hence, the equation $f(x) = 0$ cannot have any positive solution if $a \leq 0$.

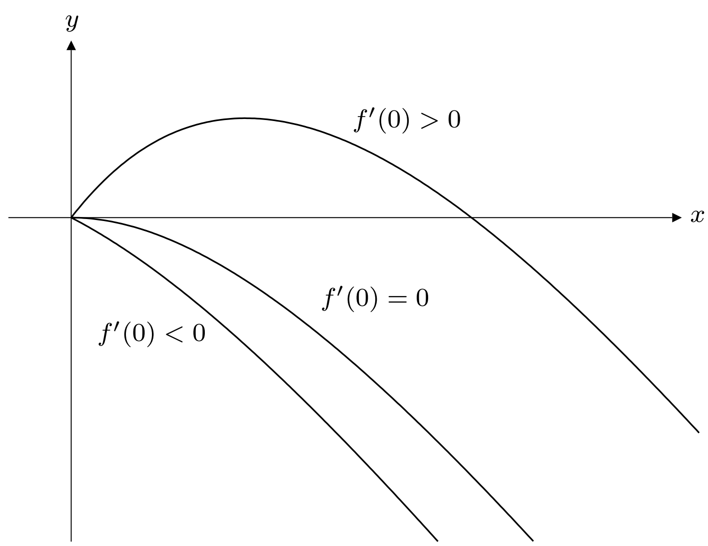
We could also argue as follows: If $x > 0$, then $e^{-x} < 1$, and we have

$$f(x) = 0 \iff \frac{4a}{1+a^2} = \frac{x}{1-e^{-x}} \iff 4a = (1+a^2)\frac{x}{1-e^{-x}}.$$

The expression on the right-hand side of the last equation is positive, and therefore a must also be positive.

Although the condition $a > 0$ is *necessary* for $f(x) = 0$ to have a positive solution, it is far from *sufficient*. We shall see below that positive solutions exist only when a lies in a certain interval.

If $a > 0$, then $f''(x) < 0$ for all x. Then $f'(x)$ is strictly decreasing and f is strictly concave. We also know that $f(x)$ will be < 0 for x sufficiently large. Thus, when $a > 0$, the equation $f(x) = 0$ has a solution $x_0 > 0$ if and only if $f'(0) > 0$ (see the figure).



Possible behaviors of $f(x)$ when $a > 0$.

We have

$$
\begin{aligned}
f'(0) > 0 &\iff \frac{4a}{1+a^2} > 1 \iff 4a > 1 + a^2 \iff a^2 - 4a + 4 < 3 \\
&\iff (a-2)^2 < 3 \iff -\sqrt{3} < a - 2 < \sqrt{3} \\
&\iff 2 - \sqrt{3} < a < 2 + \sqrt{3}.
\end{aligned}
$$

Hence, the equation $f(x) = 0$ has a positive solution if and only if a belongs to the interval $(2 - \sqrt{3}, 2 + \sqrt{3}\,)$.

(c) If $a \le 0$, then $f(x) < 0 = f(0)$ for all $x > 0$, so $f(x)$ attains its largest value for $x = x_1 = 0$.

If $a > 0$, then f is strictly concave and the following holds:

(1) If $f'(0) \le 0$, then f is strictly decreasing in all of $[0, \infty)$, and the maximum point is $x_1 = 0$.

(2) If $f'(0) > 0$, then f will first increase, and then decrease (remember that $f(x) \to -\infty$ as $x \to \infty$). Then $f(x)$ will attain its largest value at $x = x_1$, where x_1 is the unique stationary point of f. We have

$$
f'(x_1) = 0 \iff e^{x_1} = \frac{4a}{1+a^2} \iff x_1 = \ln \frac{4a}{1+a^2}\,.
$$

Conclusion:

$$
x_1 = \begin{cases} \ln \dfrac{4a}{1+a^2} & \text{if } 2 - \sqrt{3} < a < 2 + \sqrt{3}\,, \\ 0 & \text{otherwise.} \end{cases}
$$

(We found in part (b) that $f'(0) > 0$ if and only if $2 - \sqrt{3} < a < 2 + \sqrt{3}$.)

(d) There are two possibilities to be considered:

(1) If $x_1 = 0$, then $f(x_1) = f(0) = 0$ and $e^{x_1} - 1 - x_1 = e^0 - 1 - 0 = 0$.

(2) If $x_1 > 0$, then $x_1 = \ln \dfrac{4a}{1+a^2}$, and we get

$$
f(x_1) = \frac{4a}{1+a^2}(1 - e^{-x_1}) - x_1 = e^{x_1}(1 - e^{-x_1}) - x_1 = e^{x_1} - 1 - x_1.
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27 papers were handed in and were graded on a scale from 0 to 100. The highest score was 83 and the lowest was 15. The average was 58.6 points. A score of 45 points or less must be considered as very weak.