University of Oslo
Department of Economics
Arne Strøm

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Answers to some old exam problems

Exam problem 39

(a) The stationary points (x, y) for f are the solutions of the equations

$$f'_1(x,y) = \frac{1}{x+y} - 2x + 1 = 0 \iff \frac{1}{x+y} = 2x - 1 \tag{1}$$

$$f'_2(x,y) = \frac{1}{x+y} - 2y \qquad = 0 \iff \frac{1}{x+y} = 2y \tag{2}$$

We see that we must have $2y = 2x - 1$, so

$$y = x - \frac{1}{2}\,. \tag{3}$$

If we substitute this expression for y in (1) or (2), we get the equation

$$\frac{1}{2x - \frac{1}{2}} = 2x - 1.$$

Further,

$$1 = (2x - \frac{1}{2})(2x - 1) = 4x^2 - x - 2x + \frac{1}{2},$$

that is,

$$4x^2 - 3x - \frac{1}{2} = 0.$$

The roots of this quadratic equation are

$$x = \frac{3 \pm \sqrt{9 - 4 \cdot 4(-\frac{1}{2})}}{8} = \frac{3 \pm \sqrt{17}}{8}\,. \tag{4}$$

The domain of f is given as that part of the xy-plane where x and y are positive, so only the $+$ sign can be used in (4). If we then use equation (3) to determine y, we find that f has only one stationary point, namely

$$(x_0, y_0) = \Big(\frac{3+\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}\Big).$$

(b) The only stationary point for f is the point (x_0, y_0) that we found in part (a). This is then the only possible extreme point for f. The second-order partial derivatives of f are

$$f''_{11}(x,y) = -\frac{1}{(x+y)^2} - 2 = f''_{22}(x,y), \qquad f''_{12}(x,y) = -\frac{1}{(x+y)^2}\,.$$

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This yields

$$f''_{11}f''_{22} - (f''_{12})^2 = \cdots = \frac{4}{(x+y)^2} + 4 > 0$$

for all (x, y) in the domain of f. Since $f''_{11} < 0$ and $f''_{22} < 0$ everywhere, it follows from Theorem 13.1.2 in EMEA (13.1.1 in MA I), that (x_0, y_0) is a global maximum point for f.

Exam problem 51

(a) Define the Lagrangian ("Lagrange-funksjonen")

$$\mathcal{L}(x, y, z) = f(x, y, z) - \lambda(g(x, y, z) - 0) = 4z - x^2 - y^2 - z^2 - \lambda(z - xy)$$

The first-order conditions (Lagrange conditions) for (x, y, z) to be a maximum point are

$$\begin{array}{lll} (1) & (\mathcal{L}'_1(x, y, z) =) & -2x + \lambda y = 0 \\ (2) & (\mathcal{L}'_2(x, y, z) =) & -2y + \lambda x = 0 \\ (3) & (\mathcal{L}'_3(x, y, z) =) & 4 - 2z - \lambda = 0 \\ (4) & & z - xy = 0 \end{array}$$

(b)

(I) Suppose that $x = 0$. Then (2) and (4) show that $y = z = 0$ as well, and (3) gives $\lambda = 4$.

(II) If $x \neq 0$, then $y \neq 0$ because of (1). It follows from (1) that $x = \frac{1}{2}\lambda y$. Equation (2) then gives $y = \frac{1}{2}\lambda x = \frac{1}{4}\lambda^2 y$, so $\lambda^2 = 4$, that is, $\lambda = \pm 2$. It then follows from (3) that $z = 2 - \frac{1}{2}\lambda$.

(IIa) For $\lambda = 2$, we get $x = y$ and $z = 1$. Furthermore, $z = xy = x^2$, so $x = y = \pm 1$.

(IIb) For $\lambda = -2$, we get $x = -y$ and $z = 3$. But that gives $x^2 = -xy = -z = -3$, which is impossible.

Altogether we have the following three solutions of the first-order conditions:

$$\begin{aligned} (x_1, y_1, z_1, \lambda_1) &= (0, 0, 0, 4), \\ (x_2, y_2, z_2, \lambda_2) &= (1, 1, 1, 2), \\ (x_3, y_3, z_3, \lambda_3) &= (-1, -1, 1, 2). \end{aligned}$$

(c) Formula (3) on page 508 of EMEA (formula (5) on page 505 of MA I) shows that the change in the maximum value of f^* will be

$$\Delta f^* = f^*(c + \Delta c) - f^*(c) \approx df^*(c) = \lambda_2 \cdot \Delta c = 2 \cdot 0.1 = 0.2.$$

(We write the constraint ("bibetingelsen") as $z - xy = c$ and increase c by $\Delta c = 0.1$ from 0 to 0.1.)

Exam problem 73

With the Lagrangian

$$\mathcal{L}(x,y,z) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 + 4z^2 - 1) - \mu(x + 3y + 2z)$$

the necessary first-order conditions for maximum are

$$\begin{array}{llr} (1) & (\mathcal{L}'_1(x,y,z) =) & 2x - 2\lambda x - \mu = 0 \\ (2) & (\mathcal{L}'_2(x,y,z) =) & 2y - 2\lambda y - 3\mu = 0 \\ (3) & (\mathcal{L}'_3(x,y,z) =) & 2z - 8\lambda z - 2\mu = 0 \end{array}$$

together with the constraints

$$\begin{array}{lr} (4) & x^2 + y^2 + 4z^2 = 1 \\ (5) & x + 3y + 2z = 0 \end{array}$$

Equation (1) gives

$$(6) \qquad \mu = 2x - 2\lambda x = 2(1-\lambda)x.$$

We substitute this expression for μ in (2), and get

$$2(1-\lambda)y - 6(1-\lambda)x = 0 \iff 2(1-\lambda)(y - 3x) = 0.$$

Hence, $\lambda = 1$ or $y = 3x$ (or both).

A. Suppose $\underline{\lambda = 1}$. Then (6) gives $\underline{\mu = 0}$, and (3) gives $2z - 8z = 0$, that is, $z = 0$. It then follows from (5) that $x = -3y$, and equation (4) gives $9y^2 + y^2 = 1$, so $y = \pm\sqrt{1/10} = \pm 1/\sqrt{10}$.

This leads to two solutions of the first-order equations:

$$(x_1, y_1, z_1) = \Big(-\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}, 0\Big), \qquad (x_2, y_2, z_2) = \Big(\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}, 0\Big).$$

B. Now assume that $\underline{\lambda \neq 1}$. Then $y = 3x$. Equation (5) gives $2z = -x - 3y = -10x$, so $z = -5x$. If we use this in (4), we get

$$x^2 + (3x)^2 + 4(-5x)^2 = 1 \iff x^2 + 9x^2 + 100x^2 = 1 \iff x = \pm\frac{1}{\sqrt{110}}.$$

This gives us the two points

$$(x_3, y_3, z_3) = \big(\frac{1}{\sqrt{110}}, \frac{3}{\sqrt{110}}, -\frac{5}{\sqrt{110}}\big),$$
$$(x_4, y_4, z_4) = \big(-\frac{1}{\sqrt{110}}, \frac{-3}{\sqrt{110}}, \frac{5}{\sqrt{110}}\big).$$

The corresponding values of λ and μ can be found as follows: With $z = -5x$, equations (1) and (3) above become

$$\begin{aligned} 2x - 2\lambda x - \mu &= 0 \\ -10x + 40\lambda x - 2\mu &= 0 \end{aligned} \quad \iff \quad \begin{aligned} 2x\lambda + \mu &= 2x \\ 20x\lambda - \mu &= 5x \end{aligned}$$

If we consider the last system as a linear equation system with λ and μ as the unknowns, it is easy to show that

$$\underline{\lambda = \frac{7}{22}} \qquad \text{and} \qquad \underline{\mu = \frac{15}{11}x = \pm\frac{15}{11\sqrt{110}}}\,.$$

Calculating the value of $f(x, y, z) = x^2 + y^2 + x^2$ at each of the four points that we have found, we get

$$\begin{aligned} f(x_1, y_1, z_1) = f(x_2, y_2, z_2) &= \frac{9}{10} + \frac{1}{10} + 0 = 1, \\ f(x_3, y_3, z_3) = f(x_4, y_4, z_4) &= \frac{1}{110} + \frac{9}{110} + \frac{25}{110} = \frac{35}{110} = \frac{7}{22}\,. \end{aligned}$$

This shows that (x_1, y_1, z_1) and (x_2, y_2, z_2) are global maximum points for f under the given constraints, *provided there is a maximum.*

How can we be sure that there is a maximum? The constraints determine a close and bounded set, and f is continuous, so the extreme value theorem ensures that f does attain both a maximum and a minimum under these constraints. It then also follows that the points (x_3, y_3, z_3) and (x_4, y_4, z_4) are minimum points.

Comment 1: Since we know that f really attains both a maximum and a minimum, it is not strictly necessary to determine the Lagrange multipliers when we look for global extreme points. All we need is to be sure that we have found all points that satisfy the Lagrange conditions. If we happen to include a few extra points, it does no harm, as these points will be exposed when we calculate the function values at all the candidate points. Think about it!

Comment 2: After all this it is almost embarrassing to point out that the whole thing would have been much easier if we had taken another look at the functions in the problem. It follows from constraint (4) that $x^2 + y^2 = 1 - 4z^2$, so the maximand, $f(x, y, z)$, equals

$$x^2 + y^2 + z^2 = (1 - 4z^2) + z^2 = 1 - 3z^2$$

throughout the admissible set. Hence, f certainly attains its maximum value at a point where $z = 0$. If we insert this value of z into (4) and (5), we find precisely the points (x_1, y_1, z_1) and (x_2, y_2, z_2) that we found above, and we know that these points must be maximum points, without having to worry about either the extreme value theorem or Lagrange's method. Oh well, that's life!

Exam problem 115

(a) With the Lagrangian

$$\mathcal{L}(x,y,z) = xy + e^z - \lambda(e^{2z} + x^2 + 4y^2 - 6)$$

we get the necessary first-order conditions

$$\begin{aligned}
&(1) \qquad & \partial\mathcal{L}/\partial x = \quad & y - 2\lambda x = 0 \\
&(2) \qquad & \partial\mathcal{L}/\partial y = \quad & x - 8\lambda y = 0 \\
&(3) \qquad & \partial\mathcal{L}/\partial z = \quad & e^z - 2\lambda e^{2z} = 0 \\
&(4) \qquad & & e^{2z} + x^2 + 4y^2 = 6
\end{aligned}$$

(Equation (4) is the constraint.)

We note that (3) implies $2\lambda e^{2z} = e^z$, which in turn yields

$$(5) \qquad \lambda = \frac{1}{2}e^{-z} > 0.$$

From (1) we get $y = 2\lambda x$, so (2) yields $x = 8\lambda(2\lambda x) = 16\lambda^2 x$, i.e.

$$(6) \qquad x(1 - 16\lambda^2) = 0$$

A. *First suppose that* $\underline{x = 0}$. Then (1) also gives $y = 0$, and (4) yields $e^{2z} = 6$, i.e. $z = \frac{1}{2}\ln 6$. This yields one solution candidate, namely

Candidate: $$(x_1, y_1, z_1) = (0, 0, \frac{1}{2}\ln 6) = (0, 0, \ln\sqrt{6}).$$

B. *Now suppose that* $\underline{x \neq 0}$. The (6) implies $\lambda^2 = 1/16$, and because (5) shows that we must have $\lambda > 0$, we get $\lambda = 1/4$. Equation (5) further yields $e^z = 1/(2\lambda) = 2$, so $\underline{z = \ln 2}$. Now the constraint yields

$$x^2 + 4y^2 = 6 - e^{2z} = 6 - 2^2 = 2.$$

From (1), $y = 2\lambda x = x/2$, and therefore $x^2 + x^2 = 2$, which yields $x^2 = 1$. Thus we find two solution candidates:

Candidates: $$(x_2, y_2, z_2) = (1, 1/2, \ln 2), \qquad (x_3, y_3, z_3) = (-1, -1/2, \ln 2).$$

Evaluation of the objective function yields

$$\begin{aligned}
f(x_1, y_1, z_1) &= 0 + e^{\ln\sqrt{6}} = \sqrt{6}, \\
f(x_2, y_2, z_2) &= 1/2 + e^{\ln 2} = 1/2 + 2 = 5/2, \\
f(x_3, y_3, z_3) &= 1/2 + e^{\ln 2} = 1/2 + 2 = 5/2.
\end{aligned}$$

Since $(5/2)^2 = 25/4 > 6$, we have $\sqrt{6} < 5/2$, and therefore the maximum is attained at the points $\underline{(x_2, y_2, z_2)}$ and $\underline{(x_3, y_3, z_3)}$.

(b) If we add a small increment Δc to the right-hand side in the constraint, then the maximum value $f^*(c)$ in problem $(*)$ increases by $\Delta f^* \approx \lambda\,\Delta c$. In our case, $\lambda = 1/4$, and with $\Delta c = 0.1$ we get

$$\Delta f^* \approx \frac{1}{4} \cdot 0.1 = 0.025.$$

(c) If we get a solution different from the one we found in (a), it must be because we get a maximum point in the open set $\{(x,y,z) : e^{2z} + x^2 + 4y^2 < 6\}$. But such a maximum point must be a stationary point for f, and no such point exists, because $f_3'(x,y,z) = e^z \neq 0$ everywhere. Hence, the new problem has the same maximum points that we found in part (a).