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Answers to the problems for 14 May

Exam problem 17

(a) Cofactor expansion along the first row yields

$$|\mathbf{D}| = a\begin{vmatrix} 14 & -15 \\ 1 & -1 \end{vmatrix} - b\begin{vmatrix} -13 & -15 \\ -1 & -1 \end{vmatrix} + c\begin{vmatrix} -13 & 14 \\ -1 & 1 \end{vmatrix} = a + 2b + c.$$

Matrix multiplication yields

$$\mathbf{C}\,\mathbf{D} = \begin{pmatrix} 1 & 3 & -7 \\ 2 & 5 & 1 \\ 1 & 2 & 7 \end{pmatrix}\begin{pmatrix} a & b & c \\ -13 & 14 & -15 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} a-32 & b+35 & c-38 \\ 2a-66 & 2b+71 & 2c-76 \\ a-33 & b+35 & c-37 \end{pmatrix}$$

If we let $a = 33$, $b = -35$, and $c = 38$, then $\mathbf{CD} = \mathbf{I}_3$, so $\mathbf{C}$ is invertible and

$$\mathbf{C}^{-1} = \mathbf{D} = \begin{pmatrix} 33 & -35 & 38 \\ -13 & 14 & -15 \\ -1 & 1 & -1 \end{pmatrix}$$

(b) Note that the determinant of $\mathbf{A}$ is $1 \cdot 2 \cdot -1 = -2 \neq 0$, so $\mathbf{A}$ is invertible. Therefore

$$\mathbf{AY} = \mathbf{CH} \iff \mathbf{Y} = \mathbf{A}^{-1}\mathbf{CH}.$$

Then, if we let $\mathbf{X} = \mathbf{C}^{-1}\mathbf{Y}$, we get

$$\mathbf{BX} = (\mathbf{C}^{-1}\mathbf{AC})(\mathbf{C}^{-1}\mathbf{Y}) = \mathbf{C}^{-1}\mathbf{AY} = \mathbf{C}^{-1}\mathbf{CH} = \mathbf{H}.$$

Exam problem 19

(a) We get

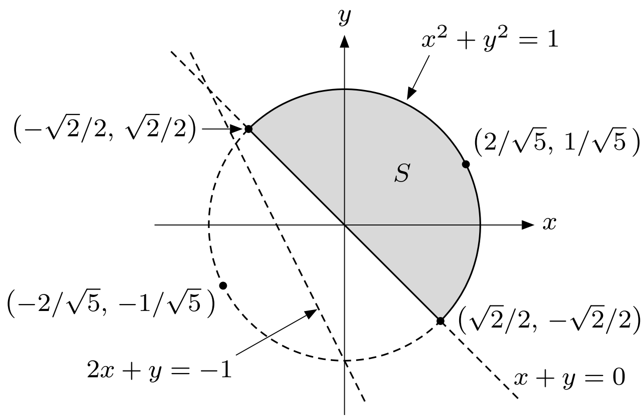
$$\frac{\partial f}{\partial x} = \frac{2}{2x+y+2} - 2, \qquad \frac{\partial f}{\partial y} = \frac{1}{2x+y+2} - 1,$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-4}{(2x+y+2)^2}, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{-2}{(2x+y+2)^2}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{-1}{(2x+y+2)^2}.$$

(b) We see that

$$\frac{\partial f}{\partial x} = 0 \iff 2x+y+2 = 1 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 0 \iff 2x+y+2 = 1$$

Mat2v09-3f 14.5.2009 715



Exam problem 19(c)

Hence, the stationary points are precisely the points that lie on the straight line $2x + y = -1$.

(c) The set S is closed and bounded and f is continuous, so by the extreme value theorem f will certainly attain a maximum over S. The stationary points of f lie on the straight line $2x + y = -1$, which does not meet S. Hence, the maximum point (or points) must lie on the boundary of S.

Along the straight part of the boundary we have $x + y = 0$, so there we have

$$f(x, y) = f(x, -x) = \ln(x + 2) - x\,.$$

Thus we need to investigate the values of $p(x) = \ln(x + 2) - x$ as x runs through the interval $\left[-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right]$. Since

$$p'(x) = \frac{1}{x + 2} - 1 < 0$$

when $x > -1$, the function p is strictly decreasing throughout the interval in question. Hence, the maximum point for f over the straight part of the boundary is $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

Along the curved part of the boundary we can use Lagrange's method to solve the problem

$$\text{maximize} \quad f(x, y) \quad \text{subject to} \quad x^2 + y^2 = 1.$$

The only point on the semicircle that satisfies the Lagrange conditions is $\left(\frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right)$. It is easy to see that

$$f\left(\frac{2}{5}\sqrt{5}, \frac{1}{5}\sqrt{5}\right) < f\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right),$$

Therefore the maximum point for f over S is $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

However, we can save ourselves the trouble of using Lagrange's method if we notice the following: At every point (x, y) in S we have $2x + y > -1$, so $2x + y + 2 > 1$, and therefore $f'_x < 0$ and $f'_y < 0$. It follows that if we move straight right or straight up from a point in S, then the value of f will decrease. It is clear from the figure that we can reach any point in S by starting at a point on the straight part of the boundary and moving up or towards the right. Hence, the maximum point for f must lie on this straight line segment, and it follows from what we saw above that the maximum point is $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$. The maximum *value* is

$$f_{\max} = f\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right) = \ln\left(2 - \frac{1}{2}\sqrt{2}\right) + \frac{1}{2}\sqrt{2}.$$

Exam problem 25

(a) We know that $\mathbf{A}_t$ has an inverse if and only if $|\mathbf{A}_t| \neq 0$. Expansion along the first row yields

$$|\mathbf{A}_t| = 1\begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix} - 0 + t\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1-t) + t \cdot 2 = 1 + t,$$

so $\mathbf{A}_t$ has an inverse if and only if $t \neq -1$.

Direct calculation gives $x + y = z$.

$$\mathbf{B}\mathbf{A}_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 1 \\ 2 & 1 & t \end{pmatrix}, \quad \mathbf{I} - \mathbf{B}\mathbf{A}_t = \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & -1 \\ -2 & -1 & 1-t \end{pmatrix}.$$

It is easy to see that $|\mathbf{I} - \mathbf{B}\mathbf{A}_t| = 0$ for all t, so $\mathbf{I} - \mathbf{B}\mathbf{A}_t$ does not have an inverse for any value of t.

(b) We simply solve the matrix equation with respect to $\mathbf{X}$:

$$\begin{aligned} \mathbf{B} + \mathbf{X}\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1} &\iff (\mathbf{B} + \mathbf{X}\mathbf{A}_1^{-1})\mathbf{A}_1 = \mathbf{A}_1^{-1}\mathbf{A}_1 \\ &\iff \mathbf{B}\mathbf{A}_1 + \mathbf{X} = \mathbf{I} \\ &\iff \mathbf{X} = \mathbf{I} - \mathbf{B}\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix} \end{aligned}$$

Exam problem 39

(a) The stationary points (x, y) for f are the solutions of the equations

(1) $$f_1'(x,y) = \frac{1}{x+y} - 2x + 1 = 0 \iff \frac{1}{x+y} = 2x - 1$$

(2) $$f_2'(x,y) = \frac{1}{x+y} - 2y = 0 \iff \frac{1}{x+y} = 2y$$

We see that we must have $2y = 2x - 1$, so

(3) $$y = x - \frac{1}{2}.$$

If we substitute this expression for y in (1) or (2), we get the equation

$$\frac{1}{2x-\frac{1}{2}} = 2x - 1.$$

Further,

$$1 = (2x - \frac{1}{2})(2x-1) = 4x^2 - x - 2x + \frac{1}{2},$$

that is,

$$4x^2 - 3x - \frac{1}{2} = 0.$$

The roots of this quadratic equation are

$$(4) \qquad x = \frac{3 \pm \sqrt{9 - 4\cdot 4(-\frac{1}{2})}}{8} = \frac{3\pm\sqrt{17}}{8}.$$

The domain of f is given as that part of the xy-plane where x and y are positive, so only the + sign can be used in (4). If we then use equation (3) to determine y, we find that f has only one stationary point, namely

$$(x_0, y_0) = \Big(\frac{3+\sqrt{17}}{8}, \frac{\sqrt{17}-1}{8}\Big).$$

(b) The only stationary point for f is the point (x_0, y_0) that we found in part (a). This is then the only possible extreme point for f. The second-order partial derivatives of f are

$$f''_{11}(x,y) = -\frac{1}{(x+y)^2} - 2 = f''_{22}(x,y), \qquad f''_{12}(x,y) = -\frac{1}{(x+y)^2}.$$

This yields

$$f''_{11}f''_{22} - (f''_{12})^2 = \cdots = \frac{4}{(x+y)^2} + 4 > 0$$

for all (x,y) in the domain of f. Since $f''_{11} < 0$ and $f''_{22} < 0$ everywhere, it follows from Theorem 13.1.2 in EMEA (13.1.1 in MA I) that (x_0, y_0) is a global maximum point for f.

Exam problem 42

(a) An elementary row operation followed by expansion along the second row gives

$$|\mathbf{A}_t| = \begin{vmatrix} t & 1 & 1 \\ t & 2 & 1 \\ 4 & t & 2 \end{vmatrix} \overset{-1}{\hookleftarrow} = \begin{vmatrix} t & 1 & 1 \\ 0 & 1 & 0 \\ 4 & t & 2 \end{vmatrix} = -0 + 1 \cdot \begin{vmatrix} t & 1 \\ 4 & 2 \end{vmatrix} - 0 = 2t - 4.$$

$\mathbf{A}_t$ has an inverse $\iff |\mathbf{A}_t| \neq 0 \iff t \neq 2$.

(b) Direct calculation shows that

$$\mathbf{A}_1 \cdot \frac{1}{2}\begin{pmatrix} -3 & 1 & 1 \\ -2 & 2 & 0 \\ 7 & -3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} -3 & 1 & 1 \\ -2 & 2 & 0 \\ 7 & -3 & -1 \end{pmatrix} = \mathbf{I}_3.$$

(c) The system can be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \text{that is,} \quad \mathbf{A}_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 & 1 & 1 \\ -2 & 2 & 0 \\ 7 & -3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5/2 \\ -1 \\ 11/2 \end{pmatrix}.$$

Exam problem 70

(a) We have

$$f(x,y) = (x^2+y^2)(xy+1) = x^3y + xy^3 + x^2 + y^2.$$

The first and second order partial derivatives of f are

$$f_1'(x,y) = 3x^2y + y^3 + 2x, \qquad f_2'(x,y) = x^3 + 3xy^2 + 2y,$$
$$f_{11}''(x,y) = 6xy + 2, \qquad f_{12}''(x,y) = 3x^2 + 3y^2, \qquad f_{22}''(x,y) = 6xy + 2.$$

(b) Stationary points are where both f_1' and f_2' are 0. It is easy to see that

$$(*) \qquad \left.\begin{aligned} f_1'(x,y) &= 0 \\ f_2'(x,y) &= 0 \end{aligned}\right\} \iff \left\{\begin{aligned} y^3 &= -x(2+3xy) \\ x^3 &= -y(2+3xy) \end{aligned}\right.$$

If we multiply the last pair of equations by y and x, respectively, we get

$$(**) \qquad y^4 = -xy(2+3xy) = x^4,$$

which gives $y^2 = x^2$, and consequently $y = \pm x$. It is clear that $(x,y) = (0,0)$ is one solution of the equations $(*)$. Are there any other stationary points? If so, we must have both $x \neq 0$ and $y \neq 0$.

Suppose first that $y = x \neq 0$. Equation $(**)$ then yields

$$x^4 = -x^2(2+3x^2) < 0,$$

But that is impossible. We are left with the possibility $y = -x \neq 0$, and we get

$$x^4 = x^2(2-3x^2) \iff x^2 = 2 - 3x^2 \iff x^2 = 1/2$$
$$\iff x = \pm\frac{1}{2}\sqrt{2},$$

which yields the two stationary points $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$ and $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$.

In order to classify the stationary points (as local maximum points, local minimum points, or saddle points) we use the second-derivative test and calculate $A = f''_{11}(x,y)$, $B = f''_{12}(x,y)$, and $C = f''_{22}(x,y)$ at each of the three stationary points. That gives the results

(x, y)	A	B	C	$AC - B^2$	Result
$(0, 0)$	2	0	2	4	Local min. point
$(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$	-1	3	-1	-8	Saddle point
$(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$	-1	3	-1	-8	Saddle point

(c) The extreme value theorem guarantees that f attains a maximum over S. A maximum point for f over S is either a stationary point for f in the interior of S or a boundary point of S. For every $a \neq 0$ we have $f(a, 0) = a^2 > f(0, 0) = 0$, and therefore $(0, 0)$ cannot be a maximum point. The other two stationary points of f lie in the interior of S if a is large enough, but then they cannot be maximum points because they are saddle points for f.

Hence, the maximum point or points must lie on the boundary of S, i.e. on the circle $x^2 + y^2 = a^2$. Along this curve we have $f(x, y) = a^2(xy + 1)$, and so we have the following problem:

$$\text{maximize } a^2(xy + 1) \quad \text{subject to} \quad x^2 + y^2 = a^2.$$

We use Lagrange's method with the Lagrangian

$$\mathcal{L}(x, y) = a^2(xy + 1) - \lambda(x^2 + y^2 - a^2).$$

The first-order conditions become

$$\begin{aligned}
(\mathcal{L}'_1(x, y) =) \quad & a^2 y - 2\lambda x = 0 \\
(\mathcal{L}'_2(x, y) =) \quad & a^2 x - 2\lambda y = 0 \\
(\text{constraint:}) \quad & x^2 + y^2 = a^2
\end{aligned}$$

We see that we must have $x \neq 0$ and $y \neq 0$, and so we get

$$\lambda = \frac{a^2 y}{2x} = \frac{a^2 x}{2y},$$

which gives $y^2 = x^2$, that is, $y = \pm x$.

Since $x^2 + y^2 = a^2$, we now get $2x^2 = a^2$, so $x = \pm a\sqrt{2}/2$, and there is a total of 4 points that satisfy the first-order conditions, namely

$$\left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right), \quad \left(\frac{a\sqrt{2}}{2}, -\frac{a\sqrt{2}}{2}\right), \quad \left(-\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}\right), \quad \left(-\frac{a\sqrt{2}}{2}, -\frac{a\sqrt{2}}{2}\right).$$

We calculate the values of f at each of these points and find that the maximum value is $f_{\max} = a^2(1 + a^2/2)$, which is attained at the first and the last point.

(The other two points give the function value $a^2(1 - a^2/2)$, and are minimum points for f *along the circle.* They will be minimum points for f over all of S if and only if $a \geq \sqrt{2}$.)