University of Oslo
Department of Economics
Arne Strøm

ECON3120/4120 Mathematics 2, spring 2009

Problem solutions for Seminar 2, 2–6 February 2009

Exam problem 11

(a) It is clear that $\lim_{x\to\infty} f(x) = 0 - 0 = 0$, but it is not quite so simple to see what happens as x tends to 0, because then each of the two fractions tends to ∞. However, if we pull them together into a single fraction, it turns out that we get an expression that we can handle by means of l'Hôpital's rule:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \text{“}\frac{0}{0}\text{”} = \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x} = \text{“}\frac{0}{0}\text{”}$$
$$= \lim_{x\to 0} \frac{e^x}{e^x + e^x + xe^x} = \frac{1}{2}\,.$$

It is best to differentiate the two terms of f separately before pulling them together into one fraction:

$$f'(x) = -\frac{1}{x^2} + \frac{e^x}{(e^x - 1)^2} = \frac{x^2e^x - (e^x - 1)^2}{x^2(e^x - 1)^2} = \frac{g(x)}{x^2(e^x - 1)^2}\,,$$

where $g(x)$ is the function we are going to investigate in part (b).

(b) We have

$$g'(x) = 2xe^x + x^2e^x - 2(e^x - 1)e^x = e^x h(x),$$

where

$$h(x) = 2x + x^2 - 2e^x + 2.$$

The point of this factorization is that $g'(x)$ has the same sign as $h(x)$ and h is a simpler function than g'. Now, $h(0) = 0$, so *if* we can show that h is strictly decreasing in $[0, \infty)$ it will immediately follow that $h(x) < 0$ for all $x > 0$.

Differentiation gives $h'(x) = 2 + 2x - 2e^x$, and we see that $h'(0) = 0$. What about the sign of $h'(x)$ if $x > 0$? Another differentiation yields $h''(x) = 2 - 2e^x$, and it is clear that $h''(x) < 0$ when $x > 0$. This implies that h' is strictly decreasing in $[0, \infty)$. Hence, $h'(x) < h'(0) = 0$ for $x > 0$. It follows that h itself is also strictly decreasing in $[0, \infty)$.

(If we look a little more at h'', we also see that $h''(x) > 0$ for $x < 0$, so h' is strictly increasing in $(-\infty, 0]$. All in all it follows that $h'(x) < h'(0) = 0$ for all $x \neq 0$. Hence, h is actually strictly decreasing over all of $(-\infty, \infty)$.)

Now that we have shown that h is strictly decreasing in $[0, \infty)$, it follows that $h(x) < 0$ for all $x > 0$, and so $g'(x) < 0$ as well.

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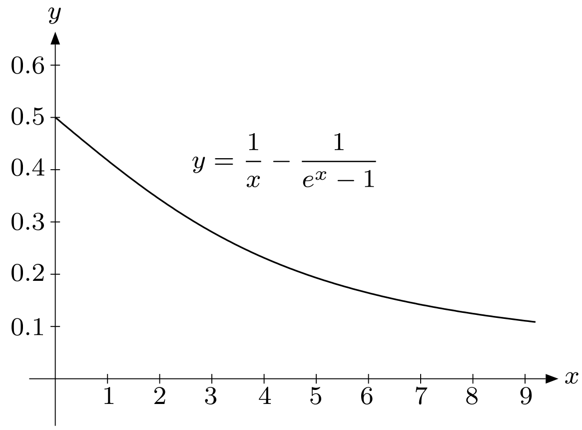
Thus, g is strictly decreasing over $[0, \infty)$, and for all $x > 0$ we then have $g(x) < g(0) = 0$.

Finally, $f'(x) = \dfrac{g(x)}{x^2(e^x - 1)^2} < 0$ for all $x > 0$, and therefore f is strictly decreasing over $[0, \infty)$.



Exam problem 11 (c).

Exam problem 22

(a) $U'(x) = aAe^{-ax} - bBe^{bx}$, $U''(x) = -a^2Ae^{-ax} - b^2Be^{bx}$.

The function U is differentiable everywhere, so any extreme point must be a stationary point.

$$\begin{aligned} U'(x) = 0 &\iff bBe^{bx} = aAe^{-ax} \iff \ln(bBe^{bx}) = \ln(aAe^{-ax}) \\ &\iff \ln(bB) + bx = \ln(aA) - ax \\ &\iff (a+b)x = \ln(aA) - \ln(bB) = \ln\Big(\frac{aA}{bB}\Big) \\ &\iff x = \frac{1}{a+b}\ln\Big(\frac{aA}{bB}\Big) = x^*. \end{aligned}$$

Hence x^* is the only stationary point of U. Moreover, $U''(x) < 0$ for all x, so $U'(x)$ is strictly decreasing everywhere. It follows that $U'(x) > 0$ for $x < x^*$ and $U'(x) < 0$ for $x > x^*$. By the first-derivative test, x^* is a (global) maximum point for U. (See p. 273 in EMEA, p. 292 in MA I.)

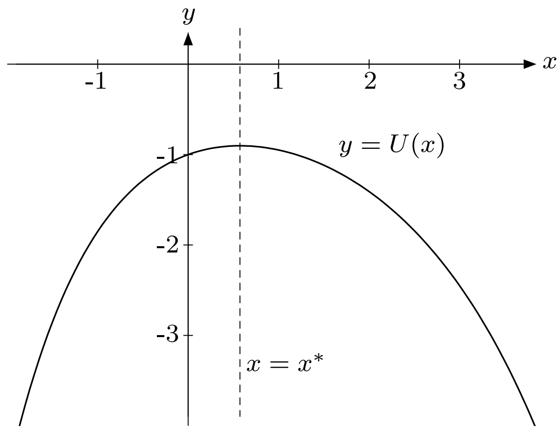
(b) It was shown in (a) that $U''(x) < 0$ for all x. Hence U is concave everywhere. The diagram shows the graph of U together with the straight line $x = x^*$ when $A = 0.6$, $B = 0.4$, $a = 1$, $b = 0.6$, and $x^* = (\ln 2.5)/1.6 \approx 0.5727$.

(c) The standard rules for powers yield

$$U(x) = -Ae^{-ax} - Be^{bx} = -Ae^{-ax^*}e^{-a(x-x^*)} - Be^{bx^*}e^{b(x-x^*)}.$$

It remains to find a C such that

$$(1)\ \ Ae^{-ax^*} = C/a \quad \text{and} \quad (2)\ \ Be^{bx^*} = C/b.$$



Exam problem 22

Equation (1) gives $C = aAe^{-ax^*}$. We know from part (a) that $U'(x^*) = 0$, and so $aAe^{-ax^*} = bBe^{bx^*}$. Hence $C/b = aAe^{-ax^*}/b = Be^{bx^*}$, i.e. equation (2) is also satisfied.

The graph of U is symmetric about the vertical line $x = x^*$ if and only if $U(x^* + t) = U(x^* - t)$ for all t. From the formula we have just shown, it follows that if $a = b$, then

$$U(x^* + t) = -\frac{C}{a}e^{-at} - \frac{C}{b}e^{bt} = -\frac{C}{a}(e^{-at} + e^{at}),$$

and so $U(x^* + t) = U(x^* + (-t)) = U(x^* - t)$.

(d) The formula for $U(x)$ in part (c) gives

$$U'(x) = Ce^{-a(x-x^*)} - Ce^{b(x-x^*)}$$
$$U''(x) = -Cae^{-a(x-x^*)} - Cbe^{b(x-x^*)}$$

Hence,

$$U(x^*) = -\frac{C}{a} - \frac{C}{b} = -C\Big(\frac{1}{a} + \frac{1}{b}\Big),$$
$$U'(x^*) = 0 \quad \text{(we already found that in part (a))},$$
$$U''(x^*) = -C(a + b).$$

The quadratic approximation to $U(x)$ around x^* is therefore

$$U(x^*) + U'(x^*)(x - x^*) + \frac{1}{2}U''(x^*)(x - x^*)^2 = -C\Big(\frac{1}{a} + \frac{1}{b}\Big) - \frac{1}{2}C(a + b)(x - x^*)^2.$$

Exam problem 26

(a) $f(x)$ is defined if and only if $4 - x^2 \geq 0$, so $D_f = [-2, 2]$. We have

$$\begin{aligned} f(x) + f(-x) &= \frac{1}{3}x^3\sqrt{4 - x^2} + \frac{1}{3}(-x)^3\sqrt{4 - (-x)^2} \\ &= \frac{1}{3}x^3\sqrt{4 - x^2} - \frac{1}{3}x^3\sqrt{4 - x^2} = 0. \end{aligned}$$

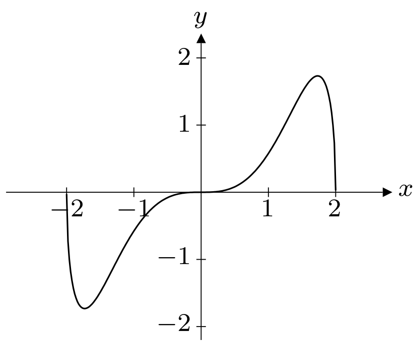
Hence, $f(-x) = -f(x)$ for all x, and so f is an *odd* function. Geometrically this means that if (x, y) lies on the graph of f, then so does $(-x, -y)$. Thus the graph is symmetric about the origin; if you rotate the graph 180° about the origin it will exactly cover the original graph. (An *even* function is a function h for which $h(-x) = h(x)$ for all x.)

(b) $$f'(x) = x^2\sqrt{4-x^2} + \frac{1}{3}x^3\frac{1}{2\sqrt{4-x^2}}(-2x) = x^2\sqrt{4-x^2} - \frac{1}{3}\frac{x^4}{\sqrt{4-x^2}}$$
$$= \frac{3x^2(4-x^2)-x^4}{3\sqrt{4-x^3}} = \frac{4x^2(3-x^2)}{3\sqrt{4-x^2}}\,.$$

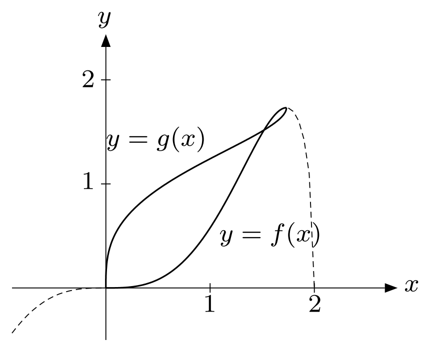
We note that

$$f'(x) < 0 \text{ if } 3 < x^2 < 4, \text{ i.e. if } \sqrt{3} < |x| < 2,$$
$$f'(x) > 0 \text{ if } 0 < x^2 < 3, \text{ i.e. if } 0 < |x| < \sqrt{3}.$$

Ït follows that f is (strictly) decreasing in $[-2, -\sqrt{3}\,]$, (strictly) increasing in $[-\sqrt{3}, \sqrt{3}\,]$, and (strictly) decreasing again in $[\sqrt{3}, 2]$. See the graph.



Exam problem 26 (c).



Exam problem 26 (d).

(d) Since f is strictly increasing in $[0, \sqrt{3}\,]$, it has an inverse function g defined on $[f(0), f(\sqrt{3}\,)] = [0, \sqrt{3}\,]$. Because g is the inverse of f we have $f(g(x)) = x$ for all x in the domain $D_g = [0, \sqrt{3}\,]$ of g. For all x in $(0, \sqrt{3}\,)$ we then have

$$\frac{d}{dx}f(g(x)) = 1\,.$$

By the chain rule, we get $f'(g(x))g'(x) = 1$. In particular, for $x = \frac{1}{3}\sqrt{3}$ we get

$$f'(g(\tfrac{1}{3}\sqrt{3}\,))g'(\tfrac{1}{3}\sqrt{3}) = 1.$$

Finally, because $f(1) = \frac{1}{3}\sqrt{3}$, we have $g(\frac{1}{3}\sqrt{3}\,) = 1$, and so

$$g'(\tfrac{1}{3}\sqrt{3}\,) = \frac{1}{f'(1)} = \frac{1}{8/(3\sqrt{3}\,)} = \frac{3\sqrt{3}}{8}\,.$$

The figure shows the graphs of f and g. Note that in this part of the problem we only consider the restriction of f to the interval $[0, \sqrt{3}\,]$. The corresponding graph of f is represented by a solid curve. The dashed parts correspond to values of x outside the interval in question.

That the domain of g is the same as the domain of (the restricted) f, is a mere accident caused by the fact that $f(0) = 0$ and $f(\frac{1}{3}\sqrt{3}\,) = \frac{1}{3}\sqrt{3}$.

Exam problem 32

(a)
$$f'(x) = \frac{(xe^{2x})' \cdot (x+1) - xe^{2x} \cdot (x+1)'}{(x+1)^2} = \frac{e^{2x}(1+2x)(x+1) - xe^{2x}}{(x+1)^2} = \frac{e^{2x}(2x^2+2x+1)}{(x+1)^2}\,.$$

The domain of f is $D_f = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)$. The function is differentiable throughout its domain, so any local extreme points must be stationary points of f. The equation $2x^2 + 2x + 1 = 0$ has no real roots, and therefore f has no local extreme points. (If we try the formula for solving quadratic equations, we get $x = \dfrac{-1 \pm \sqrt{4-8}}{4}$.)

We also have $2x^2 + 2x + 1 = x^2 + (x+1)^2 > 0$ for all x, so $f'(x) > 0$ for all $x \neq -1$. This shows that f is strictly increasing in each of the intervals $(-\infty, -1)$ and $(-1, \infty)$.

(b) It is clear that $\lim_{x\to -1} xe^{2x} = -e^{-2} < 0$. When investigating the right-hand limit $\lim_{x\to(-1)^+} f(x)$, we need to determine what happens when x is close to but greater than -1. In particular, $x+1$ will then be positive and close to 0. It follows that

$$\lim_{x\to(-1)^+} f(x) = \lim_{x\to(-1)^+} \frac{xe^{2x}}{x+1} = -\infty.$$

In a similar fashion we find that

$$\lim_{x\to(-1)^-} f(x) = \lim_{x\to(-1)^-} \frac{xe^{2x}}{x+1} = \infty,$$

since $x + 1$ is negative all the time as x tends to -1 from the left.

Further,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{x}{x+1} \cdot e^{2x}\right) = 1 \cdot 0 = 0$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{x}{x+1} \cdot e^{2x}\right) = \infty,$$

since $\lim_{x\to-\infty} x/(x+1) = \lim_{x\to\infty} x/(x+1) = 1$.

(c) The second derivative of f is

$$f''(x) = \frac{d}{dx}\left(\frac{e^{2x}(2x^2+2x+1)}{(x+1)^2}\right)$$
$$= \frac{\left[2e^{2x}(2x^2+2x+1)+e^{2x}(4x+2)\right](x+1)^2 - e^{2x}(2x^2+2x+1)2(x+1)}{(x+1)^4}$$
$$= \cdots = \frac{e^{2x}(4x^3+8x^2+8x+2)}{(x+1)^3} = \frac{e^{2x}}{(x+1)^3}g(x),$$

where $g(x) = 4x^3+8x^2+8x+2$. Then $f''(x) = 0 \iff g(x) = 0$.

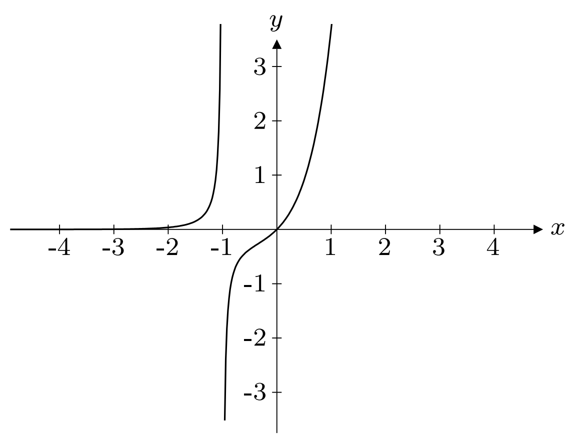
Since $g(-\frac{1}{2}) = -\frac{1}{2} < 0$ and $g(0) = 2 > 0$, there is a point x_0 in $(-\frac{1}{2}, 0)$ such that $g(x_0) = 0$. Moreover, $g'(x) = 12x^2+16x+8 = 4x^2+8(x+1)^2 > 0$ for all x. This shows that g is strictly increasing over the entire real line, and therefore $g(x) < 0$ for $x < x_0$ and $g(x) > 0$ for $x > x_0$. Hence x_0 is the only zero of g.

Since $f''(x)$ changes sign around $x = x_0$, x_0 must be an inflection point of f, and since x_0 is the only zero of f'', there are no other inflection points.

(d) The function f is convex in intervals where $f'' \geq 0$, and concave in intervals where $f'' \leq 0$. We know that $-1 < x_0$ and that

$$(x+1)^3 \begin{cases} < 0 & \text{if } x < -1, \\ > 0 & \text{if } x > -1, \end{cases} \qquad g(x) \begin{cases} < 0 & \text{if } x < x_0, \\ > 0 & \text{if } x > x_0. \end{cases}$$

A sign diagram for $f''(x) = e^{2x}g(x)/(x+1)^3$ then shows that f is convex over $(-\infty, -1)$, concave over $(-1, x_0]$ and convex again over $[x_0, \infty)$.



Exam problem 32 (d). The graph of $f(x) = \dfrac{xe^{2x}}{x+1}$.

Exam problem 105

(a) Implicit differentiation with respect to x yields

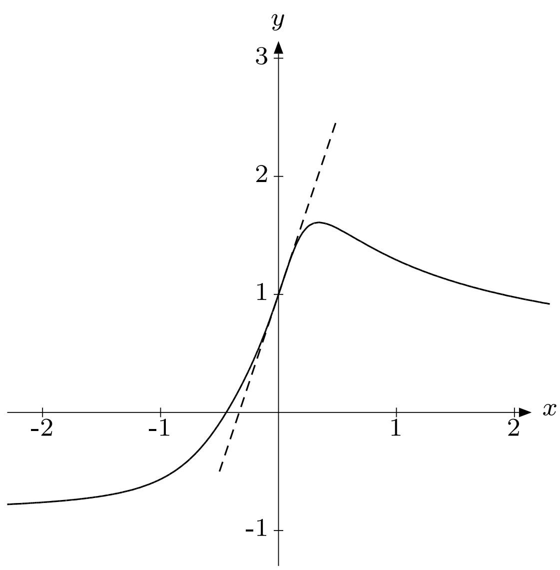
$$2xy^3 + x^2 3y^2 y' + y'e^{-x} + (y+1)(-e^{-x}) = 1$$

Letting $x = 0$, $y = 1$, we get

$$y' + 2(-1) = 1 \iff y' = 3.$$

Alternatively we can use formula (1) on page 422 in EMEA (page 424 in MA I), which yields

$$y' = -\frac{2xy^3 - (y+1)e^{-x} - 1}{3x^2y^2 + e^{-x}} = -\frac{-2-1}{1} = 3.$$



Exam problem 105

The figure shows the graph of y as a function of x, together with the tangent to the graph at $(0, 1)$.

(b) The curve intersects the x-axis when $y = 0$, that is, when

$$e^{-x} = x + 2 \qquad (\circ)$$

Let $\varphi(x) = e^{-x} - x - 2$. Then $\varphi'(x) = -e^{-x} - 1 < 0$, for all x, so $\varphi(x)$ is strictly decreasing. We have $\varphi(-1) = e - 1 > 0$ and $\varphi(0) = -1$. Hence equation $(\circ)$ has a unique solution (which lies in the interval $(-1, 0)$). This shows that the curve given by $(*)$ in the problem intersects the x-axis at exactly one place.