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ECON3120/4120 Mathematics 2, spring 2009

Problem solutions for Seminar 3, 9–13 February 2009

(For practical reasons some of the solutions may include problem parts that are not on the problem list for this seminar.)

EMEA, 7.5.5 (= MA I, 7.4.5)

We are going to need both y' and y'', so we differentiate implicitly twice in the equation
$$1 + x^3 y + x = y^{1/2}.$$
The first differentiation gives
$$3x^2 y + x^3 y' + 1 = \frac{1}{2} y^{-1/2} y'. \tag{1}$$
A second differentiation yields
$$6xy + 3x^2 y' + 3x^2 y' + x^3 y'' = -\frac{1}{4} y^{-3/2} (y')^2 + \frac{1}{2} y^{-1/2} y''. \tag{2}$$
If we now substitute $x = 0$ and $y = 1$, we get
$$(1') \quad 1 = \frac{1}{2} y' \qquad \text{and} \qquad (2') \quad 0 = -\frac{1}{4}(y')^2 + \frac{1}{2} y'',$$
which implies $y' = 2$ and $y'' = \frac{1}{2}(y')^2 = 2$ (when $x = 0$ and $y = 1$). The quadratic approximation to $y = y(x)$ is therefore
$$y(x) \approx y(0) + y'(0)x + \frac{1}{2} y''(0) x^2 = 1 + 2x + x^2.$$

Exam problem 63(a)

Implicit differentiation with respect to x in the equation $3xe^{xy^2} - 2y = 3x^2 + y^2$ gives
$$3e^{xy^2} + 3xe^{xy^2}(y^2 + 2xyy') - 2y' = 6x + 2yy'.$$
With $x = 1$ and $y = 0$, we get
$$3 - 2y'(1) = 6, \quad \text{which gives} \quad y'(1) = -3/2.$$
Hence, the slope of the graph at the point $(x^*, y^*) = (1, 0)$ is 3/2.

The linear approximation to $y(x)$ about this point is therefore
$$y(x) \approx y(1) + y'(1)(x - 1) = 0 + (-\frac{3}{2})(x - 1) = -\frac{3}{2}x + \frac{3}{2}.$$

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Problem 3

$C(x) = \int (x^2 + x - 10)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 10x + K$, where K is the constant of integration. Since $C(0) = 50$, we find that $K = 50$, so the cost function is $C(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 10x + 50$.

Problem 4

It is natural to write the integrand as a polynomial, and then

$$\int_0^2 2x^2(2-x)^2\,dx = \int_0^2 2x^2(4-4x+x^2)\,dx = \int_0^2 (2x^4 - 8x^3 + 8x^2)\,dx$$
$$= \Big|_0^2 (\frac{2}{5}x^5 - 2x^4 + \frac{8}{3}x^3) = (\frac{64}{5} - 32 + \frac{64}{3}) - 0 = \frac{32}{15} \approx 2.133.$$

The figure shows the graph of $f(x) = 2x^2(2-x)^2$ over the interval $[0, 2]$. The highest point on the graph is $B = (1, 2)$. The area between the graph and the x-axis is $\int_0^2 2x^2(2-x)^2\,dx = 32/15$. We can see from the figure that this area really is just a little bit greater than the area of the triangle OAB, which is 2.

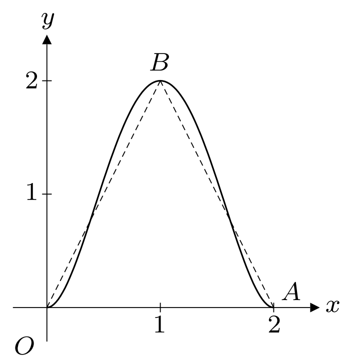

Figure for Problem 4

EMEA 9.5.1 (= MA I, 10.6.1)

(a)
$$\int \underset{\underset{f}{\uparrow}}{x}\, \underset{\underset{g'}{\uparrow}}{e^{-x}}\,dx = \underset{\underset{f}{\uparrow}}{x}\,\underset{\underset{g}{\uparrow}}{(-e^{-x})} - \int \underset{\underset{f'}{\uparrow}}{1}\cdot\underset{\underset{g}{\uparrow}}{(-e^{-x})}\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

(b)
$$\int 3xe^{4x}\,dx = 3x\cdot\frac{1}{4}e^{4x} - \int 3\cdot\frac{1}{4}e^{4x}\,dx = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$$

(c)
$$\int (1+x^2)e^{-x}\,dx = (1+x^2)(-e^{-x}) - \int 2x(-e^{-x})\,dx$$

$$= -(1+x^2)e^{-x} + 2\int xe^{-x}\,dx$$
$$= -(1+x^2)e^{-x} - 2xe^{-x} - 2e^{-x} + C \qquad \text{(use (a)!)}$$
$$= -(x^2+2x+3)e^{-x} + C$$

(d)
$$\int \underset{\substack{\uparrow\\ g'}}{x} \underset{\substack{\uparrow\\ f}}{\ln x}\,dx = \underset{\substack{\uparrow\\ g}}{\frac{x^2}{2}} \underset{\substack{\uparrow\\ f}}{\ln x} - \int \underset{\substack{\uparrow\\ g}}{\frac{x^2}{2}} \underset{\substack{\uparrow\\ f'}}{\frac{1}{x}}\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

EMEA 9.6.2 (= MA I, 10.7.2)

(b) With $u = g(x) = x^3 + 2$ we get $du = g'(x)\,dx = 3x^2\,dx$ and

$$\int x^2 e^{x^3+2}\,dx = \int e^{g(x)}\frac{1}{3}g'(x)\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{e^{x^3+2}}{3} + C\,.$$

(c) As a first attempt we could use the substitution $u = g(x) = x+2$, which gives $du = dx$ and

$$\int \frac{\ln(x+2)}{2x+4}\,dx = \int \frac{\ln u}{2u}\,du\,.$$

This does not look very much simpler than the original integral, but if we notice that $\dfrac{\ln u}{u} = \ln u \cdot \dfrac{1}{u} = \ln u \cdot \dfrac{d}{du}\ln u$, then we can see that $v = \ln u$ yields $dv = \dfrac{1}{u}\,du$ and

$$\int \frac{\ln u}{2u}\,du = \int \frac{1}{2}v\,dv = \frac{1}{4}v^2 + C = \frac{1}{4}(\ln u)^2 + C = \frac{1}{4}(\ln(x+2))^2 + C\,.$$

With a little experience we would have noticed straight away that

$$\frac{\ln(x+2)}{2x+4} = \frac{\ln(x+2)}{2(x+2)} = \frac{1}{2}\ln(x+2)\,\frac{d}{dx}\ln(x+2)\,,$$

and this immediately suggests the substitution $v = \ln(x+2)$.

Problem 7

(a) $f'(x) = (\ln x)^2 + x\big(2\ln x \cdot \dfrac{1}{x}\big) = \ln x\,(\ln x + 2), \quad f''(x) = \dfrac{2}{x}(\ln x + 1)$

(b) $f(x)$ is increasing if and only if $f'(x) = \ln x\,(\ln x + 2) \geq 0$. Note that $f'(x) = 0$ when $x = 1$ and when $\ln x = -2$, i.e. $x = e^{-2}$. A sign diagram shows that $f'(x) \geq 0$ (and $f(x)$ is increasing) if and only if $0 < x \leq e^{-2}$ or $x \geq 1$. The function is decreasing in $[e^{-2}, 1]$. $x = 1$ is a (global) minimum point since $f(x) \geq 0$ and $f(1) = 0$. Since $f(x) \to \infty$ as $x \to \infty$, there is no (global) maximum.

(c) Integration by parts, with $g'(x) = x$ and $f(x) = (\ln x)^2$, gives

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2 2(\ln x)\frac{1}{x}\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx.$$

Using the result in problem 2, we get

$$\int x(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C.$$

Problem 8

You need not look for any smart trick in order to solve the integral on the left. All you have to do is show that the derivative of the right-hand side is $\sqrt{x^2+3}$. That is pretty straightforward:

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{1}{2}x\sqrt{x^2+3} + \frac{3}{2}\ln\big(x+\sqrt{x^2+3}\big) + C\right)\\
&= \frac{1}{2}\sqrt{x^2+3} + \frac{1}{2}x\,\frac{x}{\sqrt{x^2+3}} + \frac{3}{2}\,\frac{1}{x+\sqrt{x^2+3}}\left(1+\frac{x}{\sqrt{x^2+3}}\right)\\
&= \frac{1}{2}\sqrt{x^2+3} + \frac{x^2}{2\sqrt{x^2+3}} + \frac{3}{2}\,\frac{1}{x+\sqrt{x^2+3}}\,\frac{x+\sqrt{x^2+3}}{\sqrt{x^2+3}}\\
&= \frac{1}{2}\sqrt{x^2+3} + \frac{x^2+3}{2\sqrt{x^2+3}} = \frac{1}{2}\sqrt{x^2+3} + \frac{1}{2}\sqrt{x^2+3} = \sqrt{x^2+3}\,.
\end{aligned}$$

For those of you who want to know how to find the integral: Try the substitution $u = x + \sqrt{x^2+3}$. That will give you

$$(u-x)^2 = x^2+3 \iff u^2 - 2ux = 3 \iff x = \frac{u^2-3}{2u} = \frac{u}{2} - \frac{3}{2}\cdot\frac{1}{u}\,,$$

so

$$dx = \left(\frac{1}{2} + \frac{3}{2}\cdot\frac{1}{u^2}\right)du = \frac{u^2+3}{2u^2}\,du\,.$$

Also,

$$\sqrt{x^2+3} = u - x = \frac{u}{2} + \frac{3}{2}\cdot\frac{1}{u} = \frac{u^2+3}{2u}\,.$$

Hence,

$$\int \sqrt{x^2+3}\,dx = \int \frac{u^2+3}{2u}\,\frac{u^2+3}{2u^2}\,du = \int \frac{u^4+6u^2+9}{4u^3}\,du\,.$$

This integral is easy to calculate if you write the integrand as a sum of simple fractions. Afterwards you have to substitute $x+\sqrt{x^2+3}$ for u. If you simplify the result, you will end up with the expression on the right-hand side in the problem.

(Another possibility is to substitute $x = \sqrt{3}\sinh t$, where $\sinh t = \frac{1}{2}(e^t - e^{-t})$ the hyperbolic sine function, *sinus hyperbolicus,* cf. Problem 10.8.9 in MA I.)